AD-A059 043 ARIZONA UNIV TUCSON DEPT OF MATHEMATICS F/G 12/1

AN ASYMPTOTIC THEORY FOR A CLASS OF NONLINEAR ROBIN PROBLEMS. I--ETC(U)

AUG 78 F A HOWES N00014-76-C-0326

UNCLASSIFIED TR-24 NL

| OF |

AD A059043

END

DATE FILMED

11-78

DDC

LEVEL III

(11)

9 Technical rept.,

14 TR-24

6 AN ASYMPTOTIC THEORY FOR A CLASS OF NONLINEAR ROBIN PROBLEMS. II.

by

10 F. A. Howes(*)
Department of Mathematics
University of Arizona
Tucson, Arizona 85721
(Until 15 Dec. 78)

and

School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455

11 AUG [redacted] 78

12 62p

(*)Supported in part by the National Science Foundation under Grant No. MCS 78-00907 and by the Office of Naval Research under Contract No. N00014-76-C-0326,

15 ✓NSF-MCS78-00907

LIST OF SYMBOLS

α - l.c. alpha

β - l.c. beta

γ - l.c. gamma

δ - l.c. delta

ε - l.c. epsilon

ζ - l.c. zeta

η - l.c. eta

θ - l.c. theta

λ - l.c. lambda

μ - l.c. mu

ν - l.c. nu

σ - l.c. sigma

τ - l.c. tau

ω - l.c. omega

$\mathcal{D}$ - script D

$\mathcal{F}$ - script F

$\mathcal{G}$ - script G

$\mathcal{H}$ - script H

$\mathcal{N}$ - script N

$\mathcal{R}$ - script R

$\mathbb{R}$ - real numbers

$\mathcal{O}$ - big oh Landau symbol (script O)

o - small oh Landau symbol (script o)

1. Introduction

We consider here some extensions of our results on the nonlinear Robin problem

$$\varepsilon y'' = f(t,y,y'), \qquad a < t < b,$$

$$(N) \qquad p_1 y(a,\varepsilon) - p_2 y'(a,\varepsilon) = A, \qquad q_1 y(b,\varepsilon) + q_2 y'(b,\varepsilon) = B,$$

with $f(t,y,y') = \pm\, y'^2 + h(t,y)$ published in [8]. Specifically we are interested in the existence and the asymptotic behavior (as $\varepsilon \to 0^+$) of solutions of the problem (N) whose righthand side f satisfies $f(t,y,y') = O(|y'|^n)$ as $|y'| \to \infty$ for $n > 2$. Such "superquadratic" problems have been considered by the author in [9] for functions f of the form $f(t,y,y') = h(t,y)g(t,y,y')$ where $g(t,y,y') = O(|y'|^n)$, $n > 2$, and $g \geq \nu > 0$ for all (t,y,y') of interest. However this positivity assumption on g effectively eliminates the participation of nonsingular solutions of the reduced equation $f(t,y,y') = 0$ in the asymptotic description of solutions of the problem (N) for small values of $\varepsilon > 0$. The results of [8] for the quadratic functions $f(t,y,y') = \pm y'^2 + h(t,y)$ clearly show that nonsingular solutions of $f = 0$ play an interesting and important role in analyzing how solutions of (N) behave as $\varepsilon \to 0^+$. Thus it seems of interest to us to examine similar questions in the case that $f(t,y,y') = O(|y'|^n)$ as $|y'| \to \infty$ for $n > 2$ without the restriction that $f(t,y,y') = h(t,y)g(t,y,y')$.

Such problems have not received much attention in the literature on singular perturbations apparently due to the highly nonlinear dependence of f on y'. The author's papers [8] and [9] contain the latest results on the problem (N) for the functions f discussed above as well as references to the work of others. Since the writing of [9] L. Perko [15] has examined turning point phenomena for problems related to (N) using methods developed in his previous work [12], [14].

2. A First-Order Problem

In order to discuss the problem (N) we will need some results on stability theory which are most clearly illustrated by a class of first-order problems. The theory discussed in this section is very straightforward and certainly not new (cf. [17; Chapter 1] or [3; Chapter 4]); however, we have not seen it expressed before in quite the exact form that we need for our purposes here.

Consider then the singularly perturbed initial value problem

$$(F) \qquad \varepsilon z' = f(z), \quad a < t < b, \quad z(a,\varepsilon) = z_0,$$

for finite values of a and b and for small values of $\varepsilon > 0$. If the equation $f(z) = 0$ has a solution $z = \sigma$ and if σ is stable in a sense to be made precise shortly then we anticipate that the problem (F) has a solution $z = z(t,\varepsilon)$ such that

$$(2.1) \qquad \lim_{\varepsilon \to 0^+} z(t,\varepsilon) = \sigma \quad \text{for} \quad a < t \leq b.$$

(Indeed, if $f(z_0) = 0$ then $z(t,\varepsilon) \equiv z_0$ is itself a solution.) In order that the limiting relation (2.1) hold it is enough to require that either $\sigma = z_0$ or (if $\sigma \neq z_0$)

(2.2) $$(\sigma - z_0)f(\lambda) > 0 \text{ for all } \lambda \text{ in } (\sigma,z_0] \text{ or } [z_0,\sigma).$$

This follows immediately once we make the change of variable $\tau = (t - a)\varepsilon^{-1}$, rewrite $\varepsilon z' = f(z)$ as $\frac{dz}{d\tau} = f(z)$, and note that condition (2.2) is just the condition for $z = \sigma$ to be an asymptotically stable rest point of the τ-equation (cf. [6; Chapter 3]).

Our result on (F) is contained in the following lemma.

Lemma 2.1. Assume that the equation $f(z) = 0$ has a solution $z = \sigma$ and that the function f is continuously differentiable in $[\sigma,z_0] \cup [z_0,\sigma]$. Then for all values of z_0 such that $z_0 = \sigma$ or (if $z_0 \neq \sigma$) $(\sigma - z_0)f(\lambda) > 0$ for all λ in $(\sigma,z_0]$ or $[z_0,\sigma)$ the problem (F) has a solution $z = z(t,\varepsilon)$ for each sufficiently small $\varepsilon > 0$. Moreover, for t in $[a,b]$ we have that

$$z(t,\varepsilon) = \sigma + w_L(t,\varepsilon),$$

where $w_L(a,\varepsilon) = z_0 - \sigma$ and $\lim_{\varepsilon\to 0^+} w_L(t,\varepsilon) = 0$ for $a < t \leq b$.

To illustrate the content of this lemma we discuss briefly some simple examples.

Example 2.1. The linear problem $\varepsilon z' = -kz$, $t > 0$, $z(0,\varepsilon) = z_0$, for k a positive constant, has the unique solution $z(t,\varepsilon) = z_0 e^{-kt\varepsilon^{-1}}$; and so $\lim_{\varepsilon\to 0^+} z(t,\varepsilon) = 0$ for $t > 0$ (if $z_0 = 0$ then $z(t,\varepsilon) \equiv 0$). Here $f(z) = -kz$ has the unique zero $\sigma = 0$ which is certainly stable since for $z_0 \neq 0$, $-z_0 f(\lambda) = kz_0\lambda > 0$ for λ in $(0,z_0]$ or $[z_0,0)$.

Example 2.2. A similar result holds for the nonlinear problem $\varepsilon z' = -z - z^3$, $t > 0$, $z(0,\varepsilon) = z_0$. This follows because the only real zero of $f(\sigma) = -\sigma - \sigma^3$ is $\sigma = 0$ which is stable in the sense that for $z_0 \neq 0$, $-z_0 f(\lambda) = z_0\lambda(1 + \lambda^2) > 0$ for λ in $(0,z_0]$ or $[z_0,0)$. In particular, we note that $|z(t,\varepsilon)| \leq |z_0|e^{-t\varepsilon^{-1}}$ for $t \geq 0$ since $-z - z^3 = -z(1 + z^2) \geq -z(\leq -z)$ if z is negative (positive).

Example 2.3. As our final example we consider the problem $\varepsilon z' = z^2$, $t > 0$, $z(0,\varepsilon) = z_0$. The function $f(\sigma) = \sigma^2$ has $\sigma = 0$ as its only zero and we note that (for $z_0 < 0$) $-z_0 f(\lambda) > 0$ if $z_0 \leq \lambda < 0$ while (for $z_0 > 0$) $-z_0 f(\lambda) \not> 0$ if $0 < \lambda \leq z_0$. Thus Lemma 2.1 is only applicable for $z_0 \leq 0$, in which case we have that $z(t,\varepsilon) \to 0^-$ for $t > 0$ as $\varepsilon \to 0^+$.

If $z_0 > 0$ we expect that this limiting relation will not obtain since if $z(t,\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$ then $z'(t,\varepsilon) < 0$ for $t \geq 0$; however, the equation implies that $z'(t,\varepsilon) > 0$. Indeed the exact solution of this problem is $z(t,\varepsilon) = z_0(1 - z_0 t\varepsilon^{-1})^{-1}$. Consequently, for $z_0 > 0$ this function has a vertical asymptote at $t_0 = \varepsilon z_0^{-1} > 0$, that is, for fixed $\varepsilon > 0$, $\lim_{t\to t_0^-} z(t,\varepsilon) = \infty$.

A result analogous to Lemma 2.1 is valid if the problem (F) is replaced by the problem

$$(G) \qquad \varepsilon z' = f(z), \quad a < t < b, \quad z(b,\varepsilon) = z_1.$$

This follows after replacing t by $a + b - t$ in (G) and applying Lemma 2.1 to the transformed problem. We state this result here for future reference.

Lemma 2.2. Assume that the equation $f(z) = 0$ has a solution $z = \sigma$ and that the function f is continuously differentiable in $[\sigma,z_1] \cup [z_1,\sigma]$. Then for all values of z_1 such that $z_1 = \sigma$ or (if $z_1 \neq \sigma$) $(\sigma - z_1)f(\lambda) < 0$ for all λ in $(\sigma,z_1]$ or $[z_1,\sigma)$ the problem (G) has a solution $z = z(t,\varepsilon)$ for each sufficiently small $\varepsilon > 0$. Moreover, for t in $[a,b]$ we have that

$$z(t,\varepsilon) = \sigma + w_R(t,\varepsilon),$$

where $w_R(b,\varepsilon) = z_1 - \sigma$ and $\lim_{\varepsilon\to 0^+} w_R(t,\varepsilon) = 0$ for $a \leq t < b$.

We note in passing that the more general problem $\varepsilon z' = f(z,\varepsilon)$, $a < t < b$, $z(a,\varepsilon) = z_0(\varepsilon)$, must be approached with some caution as the following examples show. Consider first the problem $\varepsilon z' = z^2 - \varepsilon^2$, $t > 0$, $z(0,\varepsilon) = z_0$. It is not difficult to see that for all values of $z_0 < -\varepsilon$ the solution $z = z(t,\varepsilon)$ of this problem satisfies $\lim_{\varepsilon\to 0^+} z(t,\varepsilon) = 0$ for $t > 0$ (cf. Example 2.3). However the solutions of the related problem $\varepsilon z' = z^2 + \varepsilon^2$, $t > 0$, $z(0,\varepsilon) = z_0$, behave entirely differently in the sense that no matter how z_0 is

chosen the solution $z(t,\varepsilon)$ has a vertical asymptote at a point $t_0 = t_0(\varepsilon) > 0$, that is, $\lim_{t \to t_0^-} z(t,\varepsilon) = \infty$. What is especially disheartening about these two examples is that although $f(z,0) = z^2$ the (formally) small term ε^2 has an order one effect on the qualitative behavior of solutions.

The results of Lemmas 2.1 and 2.2 have a direct connection with a special class of Robin problems of the form (N) and this is the content of the next section.

3. Some Special Problems

We turn now to a discussion of the problem (N) when the right hand side f has a particularly simple form, namely the problem

$$(N_1) \qquad \begin{aligned} &\varepsilon y'' = f(y'), \quad a < t < b, \\ &p_1 y(a,\varepsilon) - p_2 y'(a,\varepsilon) = A, \quad q_1 y(b,\varepsilon) + q_2 y'(b,\varepsilon) = B. \end{aligned}$$

Here the constants p_1, p_2, q_1 and q_2 are nonnegative with $p_1 + q_1 > 0$ and $p_2 + q_2 > 0$, and $f(z) = O(|z|^n)$ as $|z| \to \infty$ for $n > 2$. The results we obtain for solutions of (N_1) will turn out to be characteristic for most solutions of the general problem (N).

Suppose first that $p_1 = 0$ and $p_2 = 1$. We consider then the problem

$$(N_2) \qquad \begin{aligned} &\varepsilon y'' = f(y'), \quad a < t < b, \\ &-y'(a,\varepsilon) = A, \quad q_1 y(b,\varepsilon) + q_2 y'(b,\varepsilon) = B. \end{aligned}$$

After setting $z = y'$ and disregarding (for the moment) the boundary condition at $t = b$ we see that the problem (N_2) is precisely the initial value problem (F) of the previous section with $z_0 = -A$. Now solutions of (F) are described throughout $[a,b]$ by the stable zeros of the function f with the possible exception of a small neighborhood of the point $t = a$ (cf. Lemma 2.1). Returning to the problem (N_2) we expect that if a stable solution u of $f(u') = 0$ also satisfies the right hand boundary condition, that is, if $q_1u(b) + q_2u'(b) = B$, then the solution of (N_2) for small $\varepsilon > 0$ is represented throughout $[a,b]$ by this function u. This leads us to consider the so-called reduced problem

$$(R_R) \qquad f(u') = 0, \quad a < t < b, \quad q_1u(b) + q_2u'(b) = B,$$

and to seek solutions of (R_R) which are stable in the sense described in Lemma 2.1. The solutions of $f(u') = 0$ are clearly straight lines of slope σ where $f(\sigma) = 0$ and therefore the solution of (R_R) is $u = u_R(t) = \sigma t + c$ where $c = q_1^{-1}[B - \sigma(q_1b + q_2)]$. (Note that $q_1 > 0$ by our above assumptions since $p_1 = 0$.)

We can now state and prove an existence and estimation result for the problem (N_2).

Theorem 3.1. Assume that the reduced problem (R_R) has a solution $u = u_R(t) = \sigma_R t + c$ and that the function f is continuously differentiable in $[\sigma_R,-A] \cup [-A,\sigma_R]$. Assume also that either $\sigma_R = -A$ or (if $\sigma_R \neq -A$) $(\sigma_R + A)f(\lambda) > 0$ for all λ in $(\sigma_R,-A]$ or $[-A,\sigma_R)$. Then there exists an

$\varepsilon_0 > 0$ <u>such that the problem</u> (N_2) <u>has a unique solution</u> $y = y(t,\varepsilon)$ <u>whenever</u> $0 < \varepsilon \le \varepsilon_0$. <u>In addition, for</u> t <u>in</u> $[a,b]$ <u>we have that</u>

$$y(t,\varepsilon) = u_R(t) + O(w_L(t,\varepsilon))$$

(3.1) <u>and</u>

$$y'(t,\varepsilon) = \sigma_R + O(w_L'(t,\varepsilon)),$$

<u>where the function</u> w_L <u>is a solution of</u> $\varepsilon w_L'' = f(\sigma_R + w_L')$, $a < t < b$, $w_L'(a,\varepsilon) = -(\sigma_R + A)$, <u>satisfying</u> $\lim_{\varepsilon\to 0^+} w_L(t,\varepsilon) = 0$ <u>for</u> $a \le t \le b$ <u>and</u> $\lim_{\varepsilon\to 0^+} w_L'(t,\varepsilon) = 0$ <u>for</u> $a < t \le b$.

<u>Proof</u>. The uniqueness of y follows immediately from the maximum principle (cf. [16]). To prove the existence of a solution satisfying the limiting relations (3.1) we assume without loss of generality that $\sigma_R = 0$ (and so $u_R(t) \equiv c = q_1^{-1}B$). If $A = 0$ then $y(t,\varepsilon) \equiv 0$ (and $w_L \equiv 0$). Thus suppose that $A \neq 0$. The existence of a function w_L with the above properties follows from our stability assumption (cf. Section 2) if ε is sufficiently small, say $0 < \varepsilon \le \varepsilon_0$. In addition, if $-A < 0$ then $w_L > 0$ and if $-A > 0$ then $w_L < 0$.

Define now for t in $[a,b]$ and $0 < \varepsilon \le \varepsilon_0$

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= c + w_L(t,\varepsilon) \\ \beta(t,\varepsilon) &\equiv c \end{aligned}\right\} \text{ if } -A > 0,$$

and

$$\left.\begin{array}{l}\alpha(t,\varepsilon) \equiv c \\ \\ \beta(t,\varepsilon) = c + w_L(t,\varepsilon)\end{array}\right\} \text{ if } \ -A < 0.$$

We consider below just the case $-A < 0$ since the case $-A > 0$ is handled similarly. It is clear that $-\alpha'(a,\varepsilon) \leq A \leq -\beta'(a,\varepsilon)$, $q_1\alpha(b,\varepsilon) + q_2\alpha'(b,\varepsilon) \leq B \leq q_1\beta(b,\varepsilon) + q_2\beta'(b,\varepsilon)$, and that $\varepsilon\alpha'' \geq f(\alpha')$ and $\varepsilon\beta'' \leq f(\beta')$ for t in (a,b) and $0 < \varepsilon \leq \varepsilon_0$. If we could conclude that the problem (N_2) had a solution $y = y(t,\varepsilon)$ satisfying $\alpha(t,\varepsilon) \leq y(t,\varepsilon) \leq \beta(t,\varepsilon)$ for t in $[a,b]$ and $0 < \varepsilon \leq \varepsilon_0$ then the theorem would be proved. However such a conclusion cannot be drawn immediately here since $f(y') = O(|y'|^n)$ as $|y'| \to \infty$ for $n > 2$ (cf. [11]). What is required (cf. Heidel's theorem in [7] or [9]) is an a'priori bound on the derivative of any solution y of $\varepsilon y'' = f(y')$, $a < t < b$, satisfying $\alpha(t,\varepsilon) \leq y(t,\varepsilon) \leq \beta(t,\varepsilon)$. It will turn out (not surprisingly) that

$$(3.2) \qquad -A \leq y'(t,\varepsilon) \leq 0 \quad \text{for} \quad a \leq t \leq b,$$

and therefore the conclusion of Theorem 3.1 follows from Heidel's theorem. To verify (3.2) (and at the same time obtain a sharper estimate for $y'(t,\varepsilon)$) note first that $y'(t,\varepsilon) \leq 0$ by the maximum principle (cf. [16] or [2; Sec. 2]). In calculating a lower bound on y' we proceed indirectly by noting that for $\alpha \leq y \leq \beta$, y is a solution of the following Dirichlet problem in $(t_1,t_2) \subset (a,b)$

$$\varepsilon y'' = f(y'), \quad t_1 < t < t_2,$$

$$(H) \qquad y(t_1,\varepsilon) = c + \eta(t_1,\varepsilon), \quad y(t_2,\varepsilon) = c + \eta(t_2,\varepsilon),$$

where the positive function η is of order $O(w_L(t,\varepsilon))$ and $\eta(t_1,\varepsilon) \geq \eta(t_2,\varepsilon)$. Fix t_0 in $(a,b]$ and let $t_1 = t_0 - \delta_1$ and $t_2 = t_0$ for a small positive constant δ_1. Define now for t in $[t_1,t_2]$ and $0 < \varepsilon \leq \varepsilon_0$

$$\alpha_1(t,\varepsilon) = c + \eta(t_2,\varepsilon) - \mu(t_0 - t),$$

$$\beta_1(t,\varepsilon) = c + \eta(t_2,\varepsilon) + \mu(t_0 - t),$$

where $\mu = \mu(\varepsilon) = \delta_1^{-1}(\eta(t_0 - \delta_1,\varepsilon) - \eta(t_0,\varepsilon))$ is positive and of order $O(w_L'(t_0,\varepsilon))$. Clearly $\alpha_1(t_j,\varepsilon) \leq y(t_j,\varepsilon) \leq \beta_1(t_j,\varepsilon)$ for $j = 1, 2$ and we just have to show that $\varepsilon\alpha_1'' \geq f(\alpha_1')$ and $\varepsilon\beta_1'' \leq f(\beta_1')$, that is, $f(\alpha_1') \leq 0 \leq f(\beta_1')$. However these inequalities follow directly from our stability assumption for $f(\alpha_1') = f(\mu) < 0 < f(-\mu) = f(\beta_1')$ since $\mu > 0$. Therefore the function y (which is a solution of the problem (H) with $t_1 = t_0 - \delta_1$ and $t_2 = t_0$) satisfies $\alpha_1 \leq y \leq \beta_1$, that is, $|y(t,\varepsilon) - c_1| \leq \mu(t_0 - t)$ for $t_0 - \delta_1 \leq t \leq t_0$ and $c_1 = c + \eta(t_0,\varepsilon)$. We conclude directly that $|y'(t_0^-,\varepsilon)| \leq \mu$, so that in particular $|y'(t_0,\varepsilon)| \leq \mu$. Thus for each t in $(a,b]$, $|y'(t,\varepsilon)| \leq \mu(t,\varepsilon)$ where $\mu(t,\varepsilon) = O(|w_L'(t,\varepsilon)|)$. Finally we have that $y'(a,\varepsilon) \geq -A$ since $c \leq y(t,\varepsilon) \leq c + w_L(t,\varepsilon)$ in $[a,b]$. This concludes the proof of Theorem 3.1.

We consider now the general problem (N_1) under the assumption that $p_1 \geq 0$ and $p_2 > 0$. As with the problem (N_2) we assume that the associated reduced problem (R_R) has a solution $u = u_R(t)$. If u_R is stable in a sense analogous to that described in Theorem 3.1 we expect that the problem (N_1) has a solution $y = y(t,\varepsilon)$ which is close to u_R in $[a,b]$. The precise result is the next theorem.

Theorem 3.2. Assume that the reduced problem (R_R) has a solution $u = u_R(t) = \sigma_R t + c$ and that the function f is continuously differentable in $[\sigma_R, p_2^{-1}(p_1 u_R(a) - A)] \cup [p_2^{-1}(p_1 u_R(a) - A), \sigma_R]$. Assume also that either $p_1 u_R(a) - p_2\sigma_R = A$ or (if $p_1 u_R(a) - p_2\sigma_R \neq A$) $(p_1 u_R(a) - p_2\sigma_R - A)f(\lambda) < 0$ for all λ in $(\sigma_R, p_2^{-1}(p_1 u_R(a) - A)]$ or $[p_2^{-1}(p_1 u_R(a) - A), \sigma_R)$. Then the conclusion of Theorem 3.1 is valid with the exception that the function w_L satisfies $w_L'(a,\varepsilon) = p_2^{-1}(p_1 u_R(a) - p_2\sigma_R - A)$ instead of $w_L'(a,\varepsilon) = -(\sigma_R + A)$.

Proof. This theorem is proved in exactly the same manner as Theorem 3.1. After normalizing so that $\sigma_R = 0$ simply define for $a \leq t \leq b$ and $\varepsilon > 0$ sufficiently small

$$\left.\begin{aligned} \alpha(t,\varepsilon) &\equiv c \\ \beta(t,\varepsilon) &= c + w_L(t,\varepsilon) \end{aligned}\right\} \quad \text{if} \quad p_1 u_R(a) < A,$$

and

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= c + w_L(t,\varepsilon) \\ \beta(t,\varepsilon) &\equiv c \end{aligned}\right\} \quad \text{if} \quad p_1 u_R(a) > A,$$

and proceed as before.

The basic assumption in the two previous theorems was the existence of a stable solution u of the reduced equation $f(u') = 0$ which satisfied the righthand boundary condition. We could just as well have assumed that u satisfied the lefthand boundary condition and then proceeded to impose stability conditions on it so that the result corresponding to Theorem 3.2 was valid (cf. Lemma 2.2). The appropriate reduced problem is then

$$(R_L) \qquad f(u') = 0, \quad a < t < b, \quad p_1 u(a) - p_2 u'(a) = A,$$

and the next result follows by making the change of variable $t \to a + b - t$ and applying Theorem 3.2 to the transformed problem. (Note that we now require $q_1 \geq 0$ and $q_2 > 0$.)

Theorem 3.3. *Assume that the reduced problem* (R_L) *has a solution* $u = u_L(t) = \sigma_L t + c$ *and that the function* f *is continuously differentiable in* $[\sigma_L, q_2^{-1}(B - q_1 u_L(b))] \cup [q_2^{-1}(B - q_1 u_L(b)), \sigma_L]$. *Assume also that either* $q_1 u_L(b) + q_2 \sigma_L = B$ *or (if* $q_1 u_L(b) + q_2 \sigma_L \neq B$ $(q_1 u_L(b) + q_2 \sigma_L - B) f(\lambda) < 0$ *for all* λ *in* $(\sigma_L, q_2^{-1}(B - q_1 u_L(b))]$ *or* $[q_2^{-1}(B - q_1 u_L(b)), \sigma_L)$. *Then there exists an* $\varepsilon_0 > 0$ *such that the problem* (N_1) *with* $q_2 > 0$ *has a unique solution* $y = y(t,\varepsilon)$ *whenever* $0 < \varepsilon \leq \varepsilon_0$. *In addition, for* t *in* $[a,b]$ *we have that*

$$y(t,\varepsilon) = u_L(t) + O(w_R(t,\varepsilon)),$$

$$y'(t,\varepsilon) = \sigma_L + O(w_R'(t,\varepsilon)),$$

<u>where the function</u> w_R <u>is a solution of</u> $\varepsilon w_R'' = f(\sigma_L + w_R')$, $a < t < b$, $w_R'(b,\varepsilon) = q_2^{-1}(B - q_1u_L(b) - q_2\sigma_L)$, <u>satisfying</u> $\lim_{\varepsilon\to 0^+} w_R(t,\varepsilon) = 0$ <u>for</u> $a \leq t \leq b$ <u>and</u> $\lim_{\varepsilon\to 0^+} w_R'(t,\varepsilon) = 0$ <u>for</u> $a \leq t < b$.

Up to now we have considered how solutions of the problem (N_1) can exhibit nonuniform behavior at $t = a$ or $t = b$ (that is, boundary layer behavior). Suppose though that the following situation presents itself: The reduced problems (R_L) and (R_R) have solutions $u = u_L(t) = \sigma_L t + c$ and $u = u_R(t) = \sigma_R t + c'$ $(\sigma_L \neq \sigma_R)$ which intersect at a point t_0 in (a,b), that is, $u_L(t_0) = u_R(t_0)$ and $u_L'(t_0) \neq u_R'(t_0)$. If these solutions are stable in the sense that $f'(\sigma_L) \geq 0$ and $f'(\sigma_R) \leq 0$ it is reasonable to ask under what additional conditions there exists a solution $y = y(t,\varepsilon)$ of the problem (N_1) which converges to the "angular" path $u_1(t)$ defined by $u_1(t) = u_L(t)$ for $a \leq t \leq t_0$ and $u_1(t) = u_R(t)$ for $t_0 \leq t \leq b$. Indeed, this question was answered many years ago by Haber and Levinson [5] for the Dirichlet problem (N) (that is, $p_1 = q_1 = 1$ and $p_2 = q_2 = 0$). Their result for the simpler Dirichlet problem (N_1) is that if the corresponding reduced problems (R_L) and (R_R) have such stable intersecting solutions u_L and u_R then the problem (N_1) has a solution $y = y(t,\varepsilon)$ for each sufficiently small $\varepsilon > 0$ such that $\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = u_1(t)$ for $a \leq t \leq b$, and

$$\lim_{\varepsilon\to 0^+} y'(t,\varepsilon) = \begin{cases} \sigma_L & \text{for } a \leq t < t_0, \\ \sigma_R & \text{for } t_0 < t \leq b, \end{cases}$$

provided $(\sigma_R - \sigma_L)f(\lambda) > 0$ for all λ in (σ_L, σ_R) or (σ_R, σ_L).

It is possible to state an analogous result for the Robin problem (N_1) under the additional assumption that $p_1 > 0$ and $q_1 > 0$. This is the content of the next theorem.

Theorem 3.4. *Assume that the reduced problems* (R_L) *and* (R_R) *have solutions* $u = u_L(t) = \sigma_L t + c$ *and* $u = u_R(t) = \sigma_R t + c'$ $(\sigma_L \neq \sigma_R)$ *which intersect at a point* t_0 *in* (a,b). *Assume also that the function* f *is continuously differentiable in* $[\sigma_L, \sigma_R] \cup [\sigma_R, \sigma_L]$ *and that* $(\sigma_R - \sigma_L)f(\lambda) > 0$ *for all* λ in (σ_L, σ_R) *or* (σ_R, σ_L). *Then there exists an* $\varepsilon_0 > 0$ *such that the problem* (N_1) *with* $p_1 > 0$ *and* $q_1 > 0$ *has a unique solution* $y = y(t,\varepsilon)$ *whenever* $0 < \varepsilon \leq \varepsilon_0$. *In addition, we have that*

$$y(t,\varepsilon) = u_1(t) + O(w(t,\varepsilon)) \quad \text{for} \quad a \leq t \leq b,$$

$$y'(t,\varepsilon) = \sigma_L + O(w'(t,\varepsilon)) \quad \text{for} \quad a \leq t \leq t_0,$$

and

$$y'(t,\varepsilon) = \sigma_R + O(w'(t,\varepsilon)) \quad \text{for} \quad t_0 \leq t \leq b.$$

Here the continuous function w *is a solution of*

$$\varepsilon w'' = f(\sigma_L + w'), \quad a < t < t_0, \quad w'(t_0^-,\varepsilon) = \tfrac{1}{2}(\sigma_R - \sigma_L),$$

$$\varepsilon w'' = f(\sigma_R + w'), \quad t_0 < t < b, \quad w'(t_0^+,\varepsilon) = \tfrac{1}{2}(\sigma_L - \sigma_R),$$

satisfying $\lim_{\varepsilon\to 0^+} w(t,\varepsilon) = 0$ for $a \le t \le b$ and $\lim_{\varepsilon\to 0^+} w'(t,\varepsilon) = 0$ for $a \le t < t_0$ and $t_0 < t \le b$.

Proof. This theorem is proved in essentially the same manner as Theorem 3.1. The bounding functions α and β are defined as follows:

(i) if $\sigma_L < \sigma_R$ then

$$\alpha(t,\varepsilon) = u_1(t), \quad a \le t \le b,$$

and

$$\beta(t,\varepsilon) = \begin{cases} u_L(t) + w(t,\varepsilon) + p_2 p_1^{-1} w'(a,\varepsilon), & a \le t \le t_0, \\ u_R(t) + w(t,\varepsilon) - q_2 q_1^{-1} w'(b,\varepsilon), & t_0 \le t \le b; \end{cases}$$

(ii) if $\sigma_L > \sigma_R$ then

$$\alpha(t,\varepsilon) = \begin{cases} u_L(t) + w(t,\varepsilon) + p_1 p_2^{-1} w'(a,\varepsilon), & a \le t \le t_0, \\ u_R(t) + w(t,\varepsilon) - q_2 q_1^{-1} w'(b,\varepsilon), & t_0 \le t \le b, \end{cases}$$

$$\beta(t,\varepsilon) = u_1(t), \quad a \le t \le b.$$

In case (i), for example, $\varepsilon u_L'' = f(u_L')$, $\varepsilon u_R'' = f(u_R')$ in (a,b) and $q_1 u_L(b) + q_2 \sigma_L \le B$, $p_1 u_R(a) - p_2 \sigma_R \le A$, and consequently $\alpha(t,\varepsilon) = u_1(t) = \max\{u_L(t), u_R(t)\}$ is a lower solution (cf. [11]). Moreover, with w as above, $\beta'(t_0^-,\varepsilon) = \beta'(t_0^+,\varepsilon) = \frac{1}{2}(\sigma_L + \sigma_R)$ and $\varepsilon\beta'' \le f(\beta')$ for t in $(a,t_0) \cup (t_0,b)$, that is, β is an upper solution. Finally it is easy to see

that $y'(t,\varepsilon) = \sigma_L + O(w'(t,\varepsilon))$ in $[a,t_0]$ and $y'(t,\varepsilon) = \sigma_R + O(w'(t,\varepsilon))$ in $[t_0,b]$. Thus the conclusion of the theorem follows from Heidel's theorem [7]. Case (ii) is handled similarly.

Before discussing some examples we make several remarks.

Remark 3.1. If $u = u_L(t)$ is a solution of the reduced problem (R_L) then a necessary condition that u_L be stable in the sense described in Theorem 3.3 is that $f'(\sigma_L) \geq 0$. Similarly a solution $u = u_R(t)$ of (R_R) can be stable in the sense described in Theorem 3.2 only if $f'(\sigma_R) \leq 0$.

Remark 3.2. The boundary layer functions w_L and w_R have particularly simple forms if there is a positive constant k such that $f'(\sigma_R) \leq -k < 0$ and $f'(\sigma_L) \geq k > 0$. It is not difficult to see that in the case of Theorems 3.1 and 3.2 we can set $w_L(t,\varepsilon) = \varepsilon k_1^{-1}(\sigma_R + A)e^{-k_1(t-a)\varepsilon^{-1}}$ and $w_L(t,\varepsilon)$ $= -\varepsilon k_1^{-1}p_2^{-1}(p_1 u_R(a) - p_2\sigma_R - A)e^{-k_1(t-a)\varepsilon^{-1}}$, respectively, for a positive constant $k_1 \leq k$. In the case of Theorem 3.3 we can set $w_R(t,\varepsilon)$ $= \varepsilon k_1^{-1}q_2^{-1}(B - q_1 u_L(b) - q_2\sigma_L)e^{-k_1(b-t)\varepsilon^{-1}}$.

Similarly, in the case of Theorem 3.4 if there is a positive constant k such that $f'(\sigma_L) \geq k > 0$ and $f'(\sigma_R) \leq -k < 0$ then the interior layer function w assumes a simple form. Namely we can set

$$w(t,\varepsilon) = u_L(t) + \tfrac{1}{2}\varepsilon k_1^{-1}(\sigma_R - \sigma_L)e^{k_1(t-t_0)\varepsilon^{-1}}$$

for t in $[a,t_0]$ and

$$w(t,\varepsilon) = u_R(t) + \tfrac{1}{2}\varepsilon k_1^{-1}(\sigma_R - \sigma_L)e^{-k_1(t-t_0)\varepsilon^{-1}}$$

for t in $[t_0,b]$.

Remark 3.3. The assumption regarding the positivity of p_1 and q_1 is necessary for the validity of Theorem 3.4; cf. Example 3.3 below.

Remark 3.4. There is a connection between the nonoccurrence of boundary layer behavior as described by Theorems 3.2 and 3.3 and the occurrence of interior layer behavior as described by Theorem 3.4. Suppose for simplicity that $p_1 = q_1 = p_2 = q_2 = 1$ in (N_1) and suppose that the reduced problems (R_L) and (R_R) have stable solutions $u = u_L(t) = \sigma_L t + c$ and $u = u_R(t) = \sigma_R t + c'$ with $\sigma_L < \sigma_R$. If $u_L(b) + \sigma_L < B$ and $u_R(a) - \sigma_R < A$ but $f(B - u_L(b)) < 0$ and $f(u_R(a) - A) < 0$ then Theorems 3.2 and 3.3 are inapplicable because the required inequalities are violated by such A and B. We claim that if $|u_L(\tau) - u_R(\tau)|$ is not too large for $\tau = a$ and $\tau = b$ then in fact $u_L(a) > u_R(a)$ and $u_L(b) < u_R(b)$, that is, u_L and u_R intersect at a point in (a,b). To see this, note first that for $\omega = u_R(a) - u_L(a)$, $0 > f(u_R(a) - A) = f(\sigma_L + \omega)$, and so the stability of σ_L implies that $\omega < 0$ if $|\omega|$ is not too large. Similarly, for $\nu = u_R(b) - u_L(b)$, $0 > f(B - u_L(b)) = f(\sigma_R + \nu)$, and so the stability of σ_R implies that $\nu > 0$ if $|\nu|$ is not too large. Thus there is a chance that Theorem 3.4 will apply to the functions u_L and u_R if $f(\lambda) > 0$ for λ in (σ_L,σ_R). This inequality is certainly satisfied if σ_L and σ_R are adjacent stable zeros of f.

On the other hand if $\sigma_L > \sigma_R$, $u_L(b) + \sigma_L > B$, $f(B - u_L(b)) > 0$, $u_R(a) - \sigma_R > A$ and $f(u_R(a) - A) > 0$ then it follows as before that $u_L(a) < u_R(a)$ and $u_L(b) > u_R(b)$. Consequently u_L and u_R intersect in (a,b) and we are led again to consider the possibility of a crossing as described by Theorem 3.4.

We turn now to a discussion of several examples which illustrate the theory of this section.

Example 3.1. Consider first the problem (cf. Example 2.2)

$$\text{(E1)} \quad \begin{aligned} &\varepsilon y'' = -y' - y'^3, \quad 0 < t < 1, \\ &py(0,\varepsilon) - y'(0,\varepsilon) = A, \quad y(1,\varepsilon) = B, \end{aligned}$$

for $p \geq 0$. The reduced equation $f(\sigma) = -\sigma - \sigma^3 = 0$ has $\sigma = 0$ as its only real solution and since $f'(0) = -1$ we make the corresponding reduced solution u satisfy $u(1) = B$ (cf. Remark 3.1), that is, we consider $u = u_R(t) \equiv B$. Suppose first that $p = 0$. If $A = 0$ then $y(t,\varepsilon) \equiv B$ is the solution of (E1); however, if $A \neq 0$ then $Af(\lambda) = -A\lambda(1 + \lambda^2) > 0$ for λ in $(0,-A]$ or $[-A,0)$. Consequently we deduce from Theorem 3.1 that for all A the problem (E1) has a unique solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) = B + O(\varepsilon|A|e^{-t\varepsilon^{-1}})$ in $[0,1]$. Finally if $p > 0$ then for $A = pB$ $y(t,\varepsilon) \equiv B$ is the solution of (E1); while if $A \neq pB$ then $(pB - A)f(\lambda) = -\lambda(pB - A)(1 + \lambda^2) < 0$ for λ in $(0,pB - A]$ or $[pB - A,0)$. Thus by Theorem 3.2 the problem (E1) has a unique solution $y = y(t,\varepsilon)$ for all A and B such that $y(t,\varepsilon) = B + O(\varepsilon|pB - A|e^{-t\varepsilon^{-1}})$ in $[0,1]$.

We note that the Dirichlet problem (cf. [1], [4]) $\varepsilon y'' = -y' - y'^3$, $0 < t < 1$, $y(0,\varepsilon) = A$, $y(1,\varepsilon) = B$, has <u>no</u> solution if $A \neq B$ and $\varepsilon > 0$ is sufficiently small.

<u>Example 3.2</u>. Consider next the problem

$$\varepsilon y'' = y' - y'^3, \quad 0 < t < 1,$$

(E2)

$$y(0,\varepsilon) - y'(0,\varepsilon) = A, \quad y(1,\varepsilon) + y'(1,\varepsilon) = B.$$

The reduced equation $f(u') = u' - u'^3 = 0$ has now three solutions $u_1' = 1$, $u_2' = -1$ and $u_3' = 0$ which are such that $f'(\pm 1) = -2$ and $f'(0) = 1$. Thus we make u_1 and u_2 satisfy $u_j(1) + u_j'(1) = B$ for $j = 1, 2$, that is, $u_1(t) = t + B - 2$ and $u_2(t) = -t + B + 2$, and we make u_3 satisfy $u_3(0) - u_3'(0) = A$, that is, $u_3(t) \equiv A$. Consider first u_1. If $A = B - 3$ then $u_1(0) - u_1'(0) = A$ and so $y(t,\varepsilon) = t + B - 2$ is the solution of (E2). However if $A < B - 3$ then $(u_1(0) - 1 - A)f(\lambda) = (B - 3 - A)\lambda(1 - \lambda^2) < 0$ for λ in $(1, B - 2 - A]$ and so we apply Theorem 3.2 to deduce that the problem (E2) has a unique solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) = u_1(t) + O(\frac{1}{2}\varepsilon(B - 3 - A)e^{-2t\varepsilon^{-1}})$ in $[0,1]$. Finally if $A > B - 3$ we have that $(B - 3 - A)\lambda(1 - \lambda^2) < 0$ for λ in $[B - 2 - A, 1)$ <u>provided</u> that $B - 2 - A > 0$. Again from Theorem 3.2 we deduce the existence of a unique solution $y = y(t,\varepsilon)$ of (E2) (with $B - 3 < A < B - 2$) such that in $[0,1]$, $y(t,\varepsilon) = u_1(t) + O(k_1^{-1}\varepsilon|B - 3 - A|e^{-k_1 t\varepsilon^{-1}})$ for a positive constant $k_1 < 2$.

The asymptotic behavior described by the function u_2 is clearly a reflection of that described by u_1. Therefore if $B + 3 \leq A$ the problem (E2) has a unique solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) = u_2(t) + O(\frac{1}{2}\varepsilon|B + 3 - A|e^{-2t\varepsilon^{-1}})$ in $[0,1]$. While if $B + 2 < A < B + 3$ the solution $y(t,\varepsilon)$ satisfies $y(t,\varepsilon) = u_2(t) + O(k_1^{-1}\varepsilon(B + 3 - A)e^{-k_1 t\varepsilon^{-1}})$ in $[0,1]$ for a positive constant $k_1 < 2$.

Next consider the function $u_3 \equiv A$. If $A = B$ then $y(t,\varepsilon) \equiv A$ is the solution of (E2), while if $A < B(A - B)f(\lambda) = (A - B)\lambda(1 - \lambda^2) < 0$ for λ in $(0,B - A]$ <u>if</u> $B - A < 1$, and if $A > B(A - B)f(\lambda) < 0$ for λ in $[B - A,0)$ <u>if</u> $B - A > -1$. Thus for $B - 1 < A < B + 1$ we deduce from Theorem 3.3 the existence of a unique solution $y = y(t,\varepsilon)$ of (E2) such that in $[0,1]$ $y(t,\varepsilon) = O(k_1^{-1}\varepsilon|B - A|e^{-k_1(1-t)\varepsilon^{-1}})$ for positve constant $k_1 < 1$.

Note that we have proved the existence of a solution of (E2) for all boundary values A and B except those satisfying the inequalities $B - 2 \leq A \leq B - 1$ and $B + 1 \leq A \leq B + 2$. These are precisely the boundary values for which the boundary layer behavior described by Theorems 3.2 and 3.3 is impossible. Thus (cf. Remark 3.4) we are led to consider the "angular" paths

$$u_4(t) = \begin{cases} u_3(t), & 0 \leq t \leq t_0, \\ u_1(t), & t_0 \leq t \leq 1, \end{cases} \quad \text{and} \quad u_5(t) = \begin{cases} u_3(t), & 0 \leq t \leq \tilde{t}_0, \\ u_2(t), & \tilde{t}_0 \leq t \leq 1. \end{cases}$$

It follows directly that t_0 belongs to $(0,1)$ if and only if $B - 2 < A < B - 1$ while $\tilde{t}_0$ belongs to $(0,1)$ if and only if $B + 1 < A < B + 2$. Consider first

u_4. For $\sigma_L = 0$ and $\sigma_R = 1$ we see that $(\sigma_R - \sigma_L)f(\lambda) = \lambda(1 - \lambda^2) > 0$ for λ in $(0,1)$ and so Theorem 4.1 allows us to deduce the existence of a solution $y = y(t,\varepsilon)$ of (E2) for $B - 2 < A < B - 1$ such that in $[0,1]y(t,\varepsilon) = u_4(t) + O(\frac{1}{2}k_1^{-1}\varepsilon e^{-k_1|t-t_0|\varepsilon^{-1}})$ with $0 < k_1 < 1$. Similarly in the case of u_5, for $\sigma_L = 0$ and $\sigma_R = -1$ we see that $(\sigma_R - \sigma_L)f(\lambda) = -\lambda(1 - \lambda^2) > 0$ for λ in $(-1,0)$ and so the problem (E2) for $B + 1 < A < B + 2$ has a solution $y = y(t,\varepsilon)$ such that in $[0,1]y(t,\varepsilon) = u_5(t) + O(\frac{1}{2}k_1^{-1}\varepsilon e^{-k_1|t-t_0|\varepsilon^{-1}})$.

Finally if $A = B - 2$ then it is easy to show that (E2) has a solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) \to t + B - 2$ as $\varepsilon \to 0^+$ (as expected). Similarly if $A = B - 1$ or $A = B + 1$ a solution y exists and satisfies $y(t,\varepsilon) \to 0$ as $\varepsilon \to 0^+$, while if $A = B + 2$ then $y(t,\varepsilon) \to -t + B + 2$ as $\varepsilon \to 0^+$. The convergence is of course uniform in $[0,1]$ for these choices of A and B.

Example 3.3. In this final example we illustrate the remark that Theorem 3.4 is not necessarily valid if either $p_1 = 0$ or $q_1 = 0$. The problem is

$$\text{(E3)} \qquad \varepsilon y'' = 1 - y'^4, \quad 0 < t < 1, \quad -y'(0) = 1, \quad y(1) = 0,$$

which has the unique solution $y(t,\varepsilon) = 1 - t$ for all ε. Consider however the "angular" path defined by $u_1(t) = u_L(t) = -t$ for $0 \le t \le \frac{1}{2}$ and $u_1(t) = u_R(t) = t - 1$ for $\frac{1}{2} \le t \le 1$. The functions u_L and u_R are stable in the sense that $f'(u_L') = 4 > 0$ and $f'(u_R') = -4 < 0$; moreover, $(\sigma_R - \sigma_L)f(\lambda) = 2(1 - \lambda^4) > 0$ for $|\lambda| < 1$. Nevertheless there is no solution of (E3) which is close to $u_1(t)$ in $[0,1]$.

4. The General Problem

In this section we discuss several results for the general problem

$$(N)\qquad \begin{aligned} &\varepsilon y'' = f(t,y,y'), \quad a < t < b,\\ &p_1 y(a,\varepsilon) - p_2 y'(a,\varepsilon) = A, \quad q_1 y(b,\varepsilon) + q_2 y'(b,\varepsilon) = B, \end{aligned}$$

for constants p_1, p_2, q_1 and q_2 with the same properties as in Section 3. The function f is assumed to be at least continuous for all t in $[a,b]$ and for all values of y and y' under consideration; moreover, for (t,y) in compact subsets of $[a,b] \times \mathbb{R}$, $f(t,y,y') = \mathcal{O}(|y'|^n)$ as $|y'| \to \infty$ for $n > 2$. Recalling our results in Section 3 we now define certain reduced problems whose solutions we will use to study the existence and the asymptotic behavior of solutions of (N), namely

$$(R_L)\qquad \begin{aligned} &f(t,u,u') = 0, \quad a < t < t_L \le b,\\ &p_1 u(a) - p_2 u'(a) = A, \end{aligned}$$

$$(R_R)\qquad \begin{aligned} &f(t,u,u') = 0, \quad a \le t_R < t < b,\\ &q_1 u(b) + q_2 u'(b) = B, \end{aligned}$$

and

$(\mathcal{R})$ $$f(t,u,u') = 0, \quad a < t < b.$$

Solutions of $(\mathcal{R}_L)$, $(\mathcal{R}_R)$ and $(\mathcal{R})$ will be denoted by u_L, u_R and u_I respectively.

Our experience with the simpler problem (N_1) leads us to consider only solutions of these reduced problems which are stable in senses to be stated shortly. First we need to define some regions in (t,y,y')-space. Let solutions $u = u_R(t)$ and $u = u_L(t)$ of $(\mathcal{R}_R)$ and $(\mathcal{R}_L)$ exist in $[a,b]$ and let $\omega_R = p_1 u_R(a) - p_2 u_R'(a)$ and $\omega_L = q_1 u_L(b) + q_2 u_L'(b)$. Then we define the domains $\mathcal{D}(u_R)$ and $\mathcal{D}(u_L)$ as follows: (Here and below δ, δ_1, δ_2 etc. denote small positive constants.)

$$\mathcal{D}(u_R) = \{(t,y,y'): a \le t \le b, |y - u_R(t)| \le \delta_1, |y' - u_R'(t)| \le d_R(t)\}$$

where d_R is a smooth positive function such that if $p_2 > 0$ then $p_2^{-1}|A - \omega_R| \le d_R(t) \le p_2^{-1}|A - \omega_R| + \delta_2$ for $a \le t \le a + \delta/2$ and $d_R(t) \le \delta_2$ for $a + \delta \le t \le b$, while if $p_2 = 0$ then $d_R(t) \le \delta_2$ in $[a,b]$;

$$\mathcal{D}(u_L) = \{(t,y,y'): a \le t \le b, |y - u_L(t)| \le \delta_3, |y' - u_L'(t)| \le d_L(t)\}$$

where d_L is a smooth positive function such that if $q_2 > 0$ then $q_2^{-1}|B - \omega_L| \le d_L(t) \le q_2^{-1}|B - \omega_L| + \delta_4$ for $b - \delta/2 \le t \le b$ and $d_L(t) \le \delta_4$ for $a \le t \le b - \delta$, while if $q_2 = 0$ then $d_L(t) \le \delta_4$ in $[a,b]$.

Suppose next that $u = u_I(t)$ is a solution of the reduced equation (R) in $[a,b]$. Then we define the domain $\mathcal{D}(u_I)$ as $\mathcal{D}(u_I) = \mathcal{D}(u_L) \cap \mathcal{D}(u_R)$ where in the domains $\mathcal{D}(u_L)$ and $\mathcal{D}(u_R)$ just defined u_L and u_R are replaced by u_I.

We will also consider solution paths of the form

$$u_1(t) = \begin{cases} u_L(t), & a \le t \le t_0 (< t_L), \\ u_R(t), & (t_R <) t_0 \le t \le b, \end{cases} \qquad (\text{if } \ t_L > t_R)$$

$$u_2(t) = \begin{cases} u_L(t), & a \le t \le t_1, \\ u_I(t), & t_1 \le t \le t_2 \\ u_R(t), & t_2 \le t \le b, \end{cases}$$

$$u_3(t) = \begin{cases} u_I(t), & a \le t \le t_2, \\ u_R(t), & t_2 \le t \le b, \end{cases}$$

and

$$u_4(t) = \begin{cases} u_L(t), & a \le t \le t_1, \\ u_I(t), & t_1 \le t \le b, \end{cases}$$

and so we define the following domains:

$$\mathcal{D}(u_1) = \{(t,y,y'):a \le t \le b, |y - u_1(t)| \le \delta_1, |y' - u_1'(t)| \le d_1(t)\}$$

where d_1 is a smooth positive function such that $|u_L'(t_0) - u_R'(t_0)| \le d_1(t) \le |u_L'(t_0) - u_R'(t_0)| + \delta_2$ for $t_0 - \delta/2 \le t \le t_0 + \delta/2$ and $d_1(t) \le \delta_2$ for t in $[a,t_0 - \delta] \cup [t_0 + \delta,b]$;

$$\mathcal{D}(u_2) = \{(t,y,y'):a \le t \le b, |y - u_2(t)| \le \delta_3, |y' - u_2'(t)| \le d_2(t)\}$$

where d_2 is a smooth positive function such that $|u_L'(t_1) - u_I'(t_1)| \le d_2(t) \le |u_L'(t_1) - u_I'(t_1)| + \delta_4$ for $t_1 - \delta/2 \le t \le t_1 + \delta/2$, $|u_I'(t_2) - u_R'(t_2)| \le d_2(t) \le |u_I'(t_2) - u_R'(t_2)| + \delta_4$ for $t_2 - \delta/2 \le t \le t_2 + \delta/2$, and $d_2(t) \le \delta_4$ for t in $[a,t_1 - \delta] \cup [t_1 + \delta,t_2 - \delta] \cup [t_2 + \delta,b]$;

$$\mathcal{D}(u_3) = \{(t,y,y'):a \le t \le b, |y - u_3(t)| \le \delta_5, |y' - u_3'(t)| \le d_3(t)\}$$

where d_3 is a smooth positive function such that $|u_I'(t_2) - u_R'(t_2)| \le d_3(t) \le |u_I'(t_2) - u_R'(t_2)| + \delta_6$ for $t_2 - \delta/2 \le t \le t_2 + \delta/2$, $d_3(t) \le \delta_6$ for t in $[a + \delta,t_2 - \delta] \cup [t_2 + \delta,b]$, and $d_3(t) = d_R(t)$ for t in $[a,a + \delta/2]$ with u_R replaced by u_I;

$$\mathcal{D}(u_4) = \{(t,y,y'):a \le t \le b, |y - u_4(t)| \le \delta_7, |y' - u_4'(t)| \le d_4(t)\}$$

where d_4 is a smooth positive function such that $|u_L'(t_1) - u_I'(t_1)| \le d_4(t)$

$\leq |u_L'(t_1) - u_I'(t_1)| + \delta_8$ for $t_1 - \delta/2 \leq t \leq t_1 + \delta/2$, $d_4(t) \leq \delta_8$ for t in $[a,t_1 - \delta] \cup [t_1 + \delta, b - \delta]$, and $d_4(t) = d_L(t)$ for t in $[b - \delta/2, b]$ with u_L replaced by u_I.

Finally if u is any one of the solutions or solution paths defined above then we define the domain $\mathcal{D}_\delta(u)$ as

$$\mathcal{D}_\delta(u) = \{(t,y,y'):a \leq t \leq b, |y - u(t)| \leq \delta, |y' - u'(t)| \leq \delta\}.$$

We now define the various types of stability which solutions of the reduced problems can possess. In what follows the function f is assumed to be continuously differentiable with respect to y and y' in the appropriate domain.

Definition 4.1. A solution $u = u_R(t)$ of $(\mathcal{R}_R)$ which exists in $[a,b]$ is said to be strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \leq -k < 0$ $(f_{y'} \leq 0)$ in $\mathcal{D}_\delta(u_R)$.

Definition 4.2. A solution $u = u_L(t)$ of $(\mathcal{R}_L)$ which exists in $[a,b]$ is said to be strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \geq k > 0$ $(f_{y'} \geq 0)$ in $\mathcal{D}_\delta(u_L)$.

Definition 4.3. A solution $u = u_I(t)$ of $(\mathcal{R})$ which exists in $[a,b]$ is said to be locally strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \leq -k < 0$ $(f_{y'} \leq 0)$ in $\mathcal{D}_\delta(u_I) \cap [a,a + \delta]$ if $p_1u_I(a) - p_2u_I'(a) \neq A$ with $p_2 > 0$ and $f_{y'} \geq k > 0$ $(f_{y'} \geq 0)$ in $\mathcal{D}_\delta(u_I) \cap [b - \delta,b]$ if $q_1u_I(b) + q_2u_I'(b) \neq B$ with $q_2 > 0$.

Definition 4.4. A solution path $u = u_1(t)$ with $u_L'(t_0) \neq u_R'(t_0)$ is said to be strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_1) \cap [a,t_0]$ and $f_{y'} \leq -k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_1) \cap [t_0,b]$.

Definition 4.5. A solution path $u = u_2(t)$ with $u_L'(t_1) \neq u_I'(t_1)$ and/or $u_I'(t_2) \neq u_R'(t_2)$ is said to be strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_2) \cap [t_1 - \delta,t_1]$ and $f_{y'} \leq -k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_2) \cap [t_1,t_1 + \delta]$ and/or $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_2) \cap [t_2 - \delta,t_2]$ and $f_{y'} \leq -k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_2) \cap [t_2,t_2 + \delta]$.

Definition 4.6. A solution path $u = u_3(t)$ is said to be locally strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \leq -k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_3) \cap [a,a + \delta]$; moreover, if $u_I'(t_2) \neq u_R'(t_2)$ then we require also that $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_3) \cap [t_2 - \delta,t_2]$ and $f_{y'} \leq - k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_3) \cap [t_2,t_2 + \delta]$.

Definition 4.7. A solution path $u = u_4(t)$ is said to be locally strongly (weakly) y'-stable if there is a positive constant k such that $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_4) \cap [b - \delta,b]$; moreover, if $u_L'(t_1) \neq u_I'(t_1)$ then we require also that $f_{y'} \geq k > 0$ ($f_{y'} \geq 0$) in $\mathcal{D}_\delta(u_4) \cap [t_1 - \delta,t_1]$ and $f_{y'} \leq -k < 0$ ($f_{y'} \leq 0$) in $\mathcal{D}_\delta(u_4) \cap [t_1,t_1 + \delta]$.

The final definition of stability we will need involves the partial derivative f_y and for this reason will be termed y-stability in conformity with the previous definitions of y'-stability which involve $f_{y'}$. More

general definitions of y-stability are often needed and the reader can consult [8] or [9] for such definitions.

Definition 4.8. A solution or solution path $u = u(t)$ is said to be y-stable if there is a positive constant m such that $f_y \geq m > 0$ in $\mathcal{D}_\delta(u) \cap \{y' : y' = u'(t)\}$.

Using these definitions of stability we can begin our study of the nonlinear problem (N). In the theorems below we assume without stating so each time that the function f is continuous in (t,y,y') and continuously differentiable in y and y' for all values of t, y, y' in the domain $\mathcal{D}(u)$ where u is the reduced solution under consideration. Moreover, we tacitly assume that a solution of a reduced problem (R_L), (R_R) or (R) is of class $C^{(2)}$ in its interval of existence. (With regard to the "angular" path u_1 and possibly u_2, u_3 and u_4 we assume that the functions u_L, u_R and u_I which comprise these paths are of class $C^{(2)}$ in their respective intervals of existence.)

Our first result is the analog of Theorem 3.2 of the previous section and so we assume that $p_2 > 0$.

Theorem 4.1. Assume that the reduced problem (R_R) has a solution $u = u_R(t)$ which exists in $[a,b]$ and which is strongly or weakly y'-stable and y-stable. Assume also that either $p_1u_R(a) - p_2u_R'(a) = A$ or (if $p_1u_R(a) - p_2u_R'(a) \neq A$) $(p_1u_R(a) - p_2u_R'(a) - A)f(a,u_R(a),\lambda) < 0$ for all λ in $(u_R'(a), p_2^{-1}(p_1u_R(a) - A)]$ or $[p_2^{-1}(p_1u_R(a) - A), u_R'(a))$. Then there exists an $\varepsilon_0 > 0$ such that the problem (N) with $p_2 > 0$ has a solution $y = y(t,\varepsilon)$

<u>whenever</u> $0 < \varepsilon \leq \varepsilon_0$. <u>In addition, for</u> t <u>in</u> $[a,b]$ <u>we have that</u>

$$y(t,\varepsilon) = u_R(t) + O(w_L(t,\varepsilon)) + O(\varepsilon)$$

<u>and</u>

$$y'(t,\varepsilon) = u_R'(t) + O(w_L'(t,\varepsilon)) + O(\varepsilon),$$

<u>where</u> w_L <u>satisfies</u> $w_L'(a,\varepsilon) = p_2^{-1}(p_1 u_R(a) - p_2 u_R'(a) - A)$, $\lim_{\varepsilon\to 0^+} w_L(t,\varepsilon) = 0$ <u>for</u> $a \leq t \leq b$ <u>and</u> $\lim_{\varepsilon\to 0^+} w_L'(t,\varepsilon) = 0$ <u>for</u> $a < t \leq b$.

<u>Proof</u>. Despite the general nature of the function f the proof of this theorem is essentially a repetition of the proof of Theorem 3.1. Suppose for definiteness that $p_1 u_R(a) - p_2 u_R'(a) \leq A$ and define for $a \leq t \leq b$ and $0 < \varepsilon \leq \varepsilon_0$

$$\alpha(t,\varepsilon) = u_R(t) - \varepsilon\gamma m^{-1},$$

$$\beta(t,\varepsilon) = u_R(t) + w_L(t,\varepsilon) + \varepsilon\gamma m^{-1},$$

where $\gamma > 0$ is a constant to be determined momentarily and the function $w_L > 0$ has the above properties for $0 < \varepsilon \leq \varepsilon_0$. Clearly $p_1\alpha(a,\varepsilon) - p_2\alpha'(a,\varepsilon) \leq A \leq p_1\beta(a,\varepsilon) - p_2\beta'(a,\varepsilon)$ and $q_1\alpha(b,\varepsilon) + q_2\alpha'(b,\varepsilon) \leq B \leq q_1\beta(b,\varepsilon) + q_2\beta'(b,\varepsilon)$ by our choice of w_L. It is just as easy to see that $\varepsilon\alpha'' \geq f(t,\alpha,\alpha')$ and $\varepsilon\beta'' \leq f(t,\beta,\beta')$ in (a,b) if γ is chosen properly. Since $f(t,\sigma,\sigma') = f(t,u_R,u_R') + \{f(t,\sigma,u_R') - f(t,u_R,u_R')\} + \{f(t,\sigma,\sigma') - f(t,\sigma,u_R')$ we have first that

$$\varepsilon\alpha'' - f(t,\alpha,\alpha') = \varepsilon u_R'' - f(t,u_R,u_R') + f_y(t,\xi_1,u_R')\varepsilon\gamma m^{-1}$$

$$\geq -\varepsilon M + \varepsilon\gamma \geq 0 \quad \text{if} \quad \gamma \geq M = \max|u_R''|.$$

(Here $\xi_1 = u_R + O(\varepsilon)$ is the appropriate intermediate point.) Secondly

$$f(t,\beta,\beta') - \varepsilon\beta'' = f(t,u_R,u_R') + f_y(t,\xi_2,u_R')[w_L + \varepsilon\gamma m^{-1}]$$

$$+ \{f(t,\beta,\beta') - f(t,\beta,u_R')\} - \varepsilon u_R'' - \varepsilon w_L''.$$

By the stability assumptions of the theorem the quantity $\{\cdot\}-\varepsilon w_L''$ is non-negative in $[a,a+\delta)$ and of order $\underline{o}(\nu(t,\varepsilon))$ in $[a+\delta,b]$ for $\nu(t,\varepsilon) = \max\{\varepsilon,w_L(t,\varepsilon)\}$ with t in $[a+\delta,b]$. Therefore $f(t,\beta,\beta') - \varepsilon\beta'' \geq 0$ in (a,b) for $\gamma \geq M$.

The final step in the proof consists in establishing a bound on $y'(t,\varepsilon)$ for a solution of $\varepsilon y'' = f(t,y,y')$ satisfying $\alpha \leq y \leq \beta$. However it follows directly that $y'(t,\varepsilon) = u_R'(t) + O(w_L'(t,\varepsilon)) + O(\varepsilon)$ by arguing as in the proof of Theorem 3.1 and using the y'- and y-stability of u_R. For example, if $\alpha_1(t,\varepsilon) = u_R(t) - \ell\varepsilon - \mu(t_0 - t)$ with $\ell > 0$ and $\mu > 0$ then

$$f(t,\alpha_1,\alpha_1') = f(t,u_R,u_R') - f_y(t,\xi_1,u_R')[\ell\varepsilon + \mu(t_0 - t)] + f_{y'}(t,\alpha_1,\xi_2)\mu < 0$$

since $f_y > 0$ and $f_{y'} \leq 0$ for $\xi_1 = u_R + O(\varepsilon)$ and $\xi_2 = u_R' + O(\mu)$.

Thus Theorem 4.1 follows from Heidel's theorem [7].

The result corresponding to Theorem 4.1 for the reduced solution $u = u_L(t)$ (with $q_2 > 0$) follows now by making the change of variable $t \to a + b - t$ and applying Theorem 4.1 to the transformed problem, namely

Theorem 4.2. Assume that the reduced problem (R_L) has a solution $u = u_L(t)$ which exists in $[a,b]$ and which is strongly or weakly y'-stable and y-stable. Assume also that either $q_1u_L(b) + q_2u_L'(b) = B$ or (if $q_1u_L(b) + q_2u_L'(b) \neq B$) $(q_1u_L(b) + q_2u_L'(b) - B)f(b,u_L(b),\lambda) < 0$ for all λ in $(u_L'(b), q_2^{-1}(B - q_1u_L(b))]$ or $[q_2^{-1}(B - q_1u_L(b)), u_L'(b))$. Then there exists an $\varepsilon_0 > 0$ such that the problem (N) with $q_2 > 0$ has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, for t in $[a,b]$ we have that

$$y(t,\varepsilon) = u_L(t) + O(w_R(t,\varepsilon)) + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = u_L'(t) + O(w_R'(t,\varepsilon)) + O(\varepsilon),$$

where w_R satisfies $w_R'(b,\varepsilon) = q_2^{-1}(B - q_1u_L(b) - q_2u_L'(b))$, $\lim_{\varepsilon\to 0^+} w_R(t,\varepsilon) = 0$ for $a \leq t \leq b$ and $\lim_{\varepsilon\to 0^+} w_R'(t,\varepsilon) = 0$ for $a \leq t < b$.

It is often the case with the nonlinear problems under consideration here that the reduced equation has solutions $u = u_I(t)$ which cannot be made to satisfy either boundary condition. However if u_I is locally y'-stable and y-stable then it is not unreasonable to expect that the problem (N) with $p_2 > 0$ and $q_2 > 0$ has a solution $y = y(t,\varepsilon)$ which is approximated by u_I in $[a,b]$. This follows because the nonuniform behavior of y' is confined to small neighborhoods of $t = a$ and/or $t = b$ where we have y'-stability

and because the y-stability of u_I is global. These heuristic ideas are made precise in the next theorem which can be viewed as a combination of Theorems 4.1 and 4.2; its proof is omitted.

Theorem 4.3. Assume that the reduced equation (R) has a solution $u = u_I(t)$ which is locally strongly or weakly y'-stable and y-stable. Assume also that $p_1u_I(a) - p_2u_I'(a) = A$ or $(p_1u_I(a) - p_2u_I'(a) - A)f(a,u_I(a),\lambda) < 0$ for all λ in $(u_I'(a),p_2^{-1}(p_1u_I(a) - A)]$ or $[p_2^{-1}(p_1u_I(a) - A),u_I'(a))$ and that $q_1u_I(b) + q_2u_I'(b) = B$ or $(q_1u_I(b) + q_2u_I'(b) - B)f(b,u_I(b),\lambda) < 0$ for all λ in $(u_I'(b),q_2^{-1}(B - q_1u_I(b))]$ or $[q_2^{-1}(B - q_1u_I(b)),u_I'(b))$. Then there exists an $\varepsilon_0 > 0$ such that the problem (N) with $p_2 > 0$ and $q_2 > 0$ has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, for t in $[a,b]$ we have that

$$y(t,\varepsilon) = u_I(t) + O(w_L(t,\varepsilon)) + O(w_R(t,\varepsilon)) + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = u_I'(t) + O(w_L'(t,\varepsilon)) + O(w_R'(t,\varepsilon)) + O(\varepsilon),$$

where $w_L(w_R)$ has the properties given in the conclusion of Theorem 4.1 (Theorem 4.2) with $u_R(u_L)$ replaced by u_I.

We consider next the situation in which the reduced problems (R_L) and (R_R) have solutions u_L and u_R which intersect at a point t_0 in (a,b). Later (cf. Remark 4.4) we will see that such behavior is related to the non-occurrence of the type of boundary layer behavior described in Theorems 4.1 and 4.2. Recalling Theorem 3.4 we are led to the following theorem.

Theorem 4.4. Assume that the reduced problems (R_L) and (R_R) have solutions $u = u_L(t)$ and $u = u_R(t)$ in $[a, t_L)$ and (t_R, b) respectively with $t_L > t_R$ such that $u_L(t_0) = u_R(t_0) = c$ and $\sigma_L = u_L'(t_0) \neq u_R'(t_0) = \sigma_R$ at a point t_0 in (t_R, t_L). Assume also that the path $u = u_1(t)$ is strongly or weakly y'-stable and y-stable and that $(\sigma_R - \sigma_L)f(t_0, c, \lambda) > 0$ for all λ in (σ_L, σ_R) or (σ_R, σ_L). Then there exists an $\varepsilon_0 > 0$ such that the problem (N) has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, we have that

$$y(t,\varepsilon) = u_1(t) + O(w(t,\varepsilon)) + O(\varepsilon) \quad \text{for } a \leq t \leq b,$$

$$y'(t,\varepsilon) = u_L'(t) + O(w'(t,\varepsilon)) + O(\varepsilon) \quad \text{for } a \leq t \leq t_0,$$

and

$$y'(t,\varepsilon) = u_R'(t) + O(w'(t,\varepsilon)) + O(\varepsilon) \quad \text{for } t_0 \leq t \leq b,$$

where the continuous function w satisfies $w'(t_0^-,\varepsilon) = \frac{1}{2}(\sigma_R - \sigma_L)$, $w'(t_0^+,\varepsilon) = \frac{1}{2}(\sigma_L - \sigma_R)$, $\lim_{\varepsilon \to 0^+} w(t,\varepsilon) = 0$ for $a \leq t \leq b$ and $\lim_{\varepsilon \to 0^+} w'(t,\varepsilon) = 0$ for $a \leq t < t_0$ and $t_0 < t \leq b$.

Proof. Define for $a \leq t \leq b$ and $0 < \varepsilon \leq \varepsilon_0$

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= u_1(t) - \varepsilon\gamma m^{-1} \\ \beta(t,\varepsilon) &= u_1(t) + w(t,\varepsilon) + \varepsilon\gamma m^{-1} \end{aligned}\right\} \text{ if } \sigma_L < \sigma_R$$

and

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= u_1(t) + w(t,\varepsilon) - \varepsilon\gamma m^{-1} \\ \beta(t,\varepsilon) &= u_1(t) + \varepsilon\gamma m^{-1} \end{aligned}\right\} \text{if} \quad \sigma_L > \sigma_R,$$

where w has the above properties for $0 < \varepsilon \leq \varepsilon_0$. Then one verifies easily that each of the inequalities of Heidel's theorem is valid. To obtain a bound on $y'(t,\varepsilon)$ we estimate $y'(t^-,\varepsilon)$ in $[t_0,b]$ as in Theorem 4.1 and $y'(t^+,\varepsilon)$ in $[a,t_0]$ using the y'- and y-stability of u_R and u_L respectively.

Suppose now that the reduced problems (R_L), (R) and (R_R) have solutions $u = u_L(t)$, $u = u_I(t)$ and $u = u_R(t)$ such that $u_L(t_1) = u_I(t_1)$ and $u_I(t_2) = u_R(t_2)$ at distinct points t_1 and t_2 in (a,b) with $t_1 < t_L$ and $t_R < t_2$. If $u_L'(t_1) = u_I'(t_1)$ and $u_I'(t_2) = u_R'(t_2)$ it is clear (cf. the proof of Theorem 4.4) that if the path $u = u_2(t)$ is y-stable then the problem (N) has a solution $y = y(t,\varepsilon)$ for $0 < \varepsilon \leq \varepsilon_0$ such that $y(t,\varepsilon) = u_2(t) + O(\varepsilon)$ and $y'(t,\varepsilon) = u_2'(t) + O(\varepsilon)$ for $a \leq t \leq b$. However if $u_L'(t_1) \neq u_I'(t_1)$ and/or $u_I'(t_2) \neq u_R'(t_2)$ then we have the situation described in Theorem 4.4 at $t = t_1$ and/or $t = t_2$. The proof of the following result can be patterned after the proof of Theorem 4.4.

Theorem 4.5. Assume that the reduced problems (R_L), (R) and (R_R) have solutions $u = u_L(t)$, $u = u_I(t)$ and $u = u_R(t)$ such that $u_L(t_1) = u_I(t_1) = c_1$, $u_I(t_2) = u_R(t_2) = c_2$, $\sigma_1 = u_L'(t_1) \neq u_I'(t_1) = \mu_1$ and/or $\sigma_2 = u_I'(t_2) \neq u_R'(t_2) = \mu_2$. Assume also that the path $u = u_2(t)$ is strongly or weakly y'-stable and y-stable, and that $(\mu_1 - \sigma_1)f(t_1,c_1,\lambda) > 0$ for all λ in (σ_1,μ_1) or (μ_1,σ_1) (if $\sigma_1 \neq \mu_1$) and/or $(\mu_2 - \sigma_2)f(t_2,c_2,\lambda) > 0$ for all λ in

(σ_2,μ_2) or (μ_2,σ_2) (if $\sigma_2 \neq \mu_2$). Then there exists an $\varepsilon_0 > 0$ such that the problem (N) has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, we have that

$$y(t,\varepsilon) = u_2(t) + O(w_1(t,\varepsilon)) + O(w_2(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad a \leq t \leq b,$$

$$y'(t,\varepsilon) = u_L'(t) + O(w_1'(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad a \leq t \leq t_1,$$

$$y'(t,\varepsilon) = u_I'(t) + O(w_1'(t,\varepsilon)) + O(w_2'(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad t_1 \leq t \leq t_2,$$

and

$$y'(t,\varepsilon) = u_R'(t) + O(w_2'(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad t_2 \leq t \leq b.$$

Here w_j $(j = 1, 2)$ are continuous functions satisfying $w'(t_j^-,\varepsilon) = \frac{1}{2}(\mu_j - \sigma_j)$, $w_j'(t_j^+,\varepsilon) = \frac{1}{2}(\sigma_j - \mu_j)$, $\lim_{\varepsilon\to 0^+} w_j(t,\varepsilon) = 0$ for $a \leq t \leq b$ and $\lim_{\varepsilon\to 0^+} w_j'(t,\varepsilon) = 0$ for $a \leq t < t_j$ and $t_j < t \leq b$.

We consider finally the case in which the reduced equation (R) has a solution $u = u_I(t)$ which intersects a solution u_R of (R_R) or a solution u_L of (R_L) at a point in (t_R,b) or (a,t_L). The following results can be proved by combining the arguments for proving the boundary and interior layer theorems above.

Theorem 4.6. Assume that the reduced problems (R) and (R_R) have solutions $u = u_I(t)$ and $u = u_R(t)$ such that $u_I(t_2) = u_R(t_2) = c_2$ at a point t_2 in (t_R,b). Assume also that the path $u = u_3(t)$ is locally strongly or weakly y'-stable and y-stable and that either $p_1u_I(a) - p_2u_I'(a) = A$ or (if $p_1u_I(a) - p_2u_I'(a) \neq A$) $(p_1u_I(a) - p_2u_I'(a) - A)f(a,u_I(a),\lambda) < 0$ for all λ in $(u_I'(a),p_2^{-1}(p_1u_I(a) - A)]$ or $[p_2^{-1}(p_1u_I(a) - A),u_I'(a))$. Assume finally that if $\sigma_2 = u_I'(t_2) \neq u_R'(t_2) = \mu_2$ then $(\mu_2 - \sigma_2)f(t_2,c_2,\lambda) > 0$ for λ in (σ_2,μ_2) or (μ_2,σ_2). Then there exists an $\varepsilon_0 > 0$ such that the problem (N) with $p_2 > 0$ has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, we have that

$$y(t,\varepsilon) = u_3(t) + O(w_L(t,\varepsilon)) + O(w_2(t,\varepsilon)) + O(\varepsilon) \quad \text{for } a \leq t \leq b,$$

$$y'(t,\varepsilon) = u_I'(t) + O(w_L'(t,\varepsilon)) + O(w_2'(t,\varepsilon)) + O(\varepsilon) \quad \text{for } a \leq t \leq t_2,$$

and

$$y'(t,\varepsilon) = u_R'(t) + O(w_2'(t,\varepsilon)) + O(\varepsilon) \quad \text{for } t_2 \leq t \leq b.$$

Here $w_2(w_L)$ has the properties given in the conclusion of Theorem 4.5 (Theorem 4.1 with u_R replaced by u_I).

Theorem 4.7. Assume that the reduced problems (R_L) and (R) have solutions $u = u_L(t)$ and $u = u_I(t)$ such that $u_L(t_1) = u_I(t_1) = c_1$ at a point t_1 in (a,t_L). Assume also that the path $u = u_4(t)$ is locally strongly or weakly y'- and y-stable and that either $q_1u_I(b) + q_2u_I'(b) = B$ or (if $q_1u_I(b) + q_2u_I'(b) \neq B$) $(q_1u_I(b) + q_2u_I'(b) - B)f(b,u_I(b),\lambda) < 0$ for all λ in $(u_I'(b),q_2^{-1}(B - q_1u_I(b))]$ or $[q_2^{-1}(B - q_1u_I(b)),u_I'(b))$. Assume finally that if $\sigma_1 = u_L'(t_1) \neq u_I'(t_1) = \mu_1$ then $(\mu_1 - \sigma_1)f(t_1,c_1,\lambda) > 0$ for all λ in

(σ_1, μ_1) or (μ_1, σ_1). Then there exists an $\varepsilon_0 > 0$ such that the problem (N) with $q_2 > 0$ has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, we have that

$$y(t,\varepsilon) = u_4(t) + O(w_1(t,\varepsilon)) + O(w_R(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad a \leq t \leq b,$$

$$y'(t,\varepsilon) = u_L'(t) + O(w_1'(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad a \leq t \leq t_1,$$

and

$$y'(t,\varepsilon) = u_I'(t) + O(w_1'(t,\varepsilon)) + O(w_R'(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad t_1 \leq t \leq b.$$

We close this section with several remarks.

Remark 4.1. The boundary layer functions w_L and w_R assume particularly simple forms if u_R and u_L (or u_I) respectively are strongly (or locally strongly) y'-stable. Namely we can set $w_L(t,\varepsilon) = -k_1^{-1}p_2^{-1}\varepsilon(p_1u(a) - p_2u'(a) - A)e^{-k_1(t-a)\varepsilon^{-1}}$ for $u = u_R$ or u_I and $w_R(t,\varepsilon) = k_1^{-1}q_2^{-1}\varepsilon(q_1\tilde{u}(b) + q_2\tilde{u}'(b) - B)e^{-k_1(b-t)\varepsilon^{-1}}$ for $\tilde{u} = u_L$ or u_I, where k_1 is a positive constant, $k_1 \leq k$.

Similarly, the interior layer functions w, w_1 and w_2 are of exponential type if the reduced paths are strongly y'-stable. For example, in the case of Theorem 4.4 we can define w as $w(t,\varepsilon) = \frac{1}{2}k_1^{-1}\varepsilon(\sigma_R - \sigma_L)e^{k_1(t-t_0)\varepsilon^{-1}}$ for $a \leq t \leq t_0$ and $w(t,\varepsilon) = \frac{1}{2}k_1^{-1}\varepsilon(\sigma_R - \sigma_L)e^{-k_1(t-t_0)\varepsilon^{-1}}$ for $t_0 \leq t \leq b$, where $0 < k_1 \leq k$ (cf. [17, [13]).

Remark 4.2. The y-stability of the various reduced solutions u implies that the solutions of (N) described above are locally unique in the sense that for each choice of u there is only one solution y of (N) satisfying $\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = u(t)$ in $[a,b]$. However, for a given pair of boundary values A and B the problem (N) may have more than one solution for all values of $\varepsilon > 0$ sufficiently small; cf. Example 6.3 below.

Remark 4.3. We note that it was not necessary to assume that $p_1 > 0$ and $q_1 > 0$ in the statement of Theorem 4.4 whereas these restrictions were required for the validity of Theorem 3.4. This is due to the fact that the path u_1 is assumed to be y-stable in $[a,b]$.

Remark 4.4. There is also a connection between the occurrence of interior layer behavior and the nonoccurrence of boundary layer behavior for solutions of the general problem (N) (cf. Remark 3.4). Suppose for example that the reduced problems (R_L) and (R_R) have strongly or weakly y'-stable and y-stable solutions u_L and u_R such that

$$u_L(b) + u_L'(b) < B \quad \text{and} \quad u_R(a) - u_R'(a) < A$$

(4.1) but

$$f(b,u_L(b),b - u_L(b)) < 0 \quad \text{and} \quad f(a,u_R(a),u_R(a) - A) < 0.$$

(Here we assume for simplicity that $p_1 = q_1 = p_2 = q_2 = 1$ in (N).) Then Theorems 4.1 and 4.2 are inapplicable; however, suppose that $|u_L(\tau) - u_R(\tau)|$ is not too large for $\tau = a$ or $\tau = b$. We claim that

(i) $u_L(b) < u_R(b)$ and

(ii) $u_L(a) > u_R(a)$,

that is, u_L and u_R intersect at least once in (a,b). To verify inequality (i) set $\omega = u_L(b) - u_R(b)$ and note that

$$0 > f(b,u_L(b),B - u_L(b)) = f(b,u_R(b) + \omega,u_R'(b) - \omega)$$

$$(4.2) \qquad = f_y(b,u_R(b) + \theta\omega,u_R'(b) - \omega)\omega$$

$$+ f(b,u_R(b),u_R'(b) - \omega).$$

Suppose on the contrary that $u_L(b) > u_R(b)$, that is, $\omega > 0$. Then the y'-stability of u_R implies that $f(b,u_R(b),u_R'(b) - \omega) \geq 0$ while the positivity of f_y implies that $f_y\omega > 0$. Thus $f_y\omega + f(b,u_R(b),u_R'(b) - \omega) > 0$ which contradicts (4.2). Similarly, to verify inequality (ii) set $\nu = u_R(a) - u_L(a)$, then we have that

$$0 > f(a,u_R(a),u_R(a) - A) = f(a,u_L(a) + \nu,u_L'(a) + \nu)$$

$$(4.3) \qquad = f_y(a,u_L(a) + \theta\nu,u_L'(a) + \nu)\nu$$

$$+ f(a,u_L(a),u_L'(a) + \nu).$$

Suppose on the contrary that $u_L(a) < u_R(a)$, that is, $\nu > 0$. Then $f(a,u_L(a),u_L'(a) + \nu) \geq 0$ by the y'-stability of u_L while $f_y\nu > 0$ by the positivity of f_y. Consequently $f_y\nu + f(a,u_L(a),u_L'(a) + \nu) > 0$, which contradicts (4.3).

Thus when the inequalities (4.1) obtain the functions u_L and u_R intersect in (a,b) and we can check further to see if Theorem 4.4 is applicable. Such will always be the case if $u_L'(t_0)$ and $u_R'(t_0)$ $(u_L(t_0) = u_R(t_0) = c)$ are adjacent zeros of $f(t_0,c,\sigma)$. Likewise if $u_L(b) + u_L'(b) > B$ and $u_R(a) - u_R'(a) > A$ but $f(b,u_L(b),B - u_L(b)) > 0$ and $f(a,u_R(a),u_R(a) - A) > 0$ then u_L and u_R intersect at least once in (a,b) and there is the possibility of a crossing as described by Theorem 4.4.

5. Some Singular Phenomena

The results of the previous section are distinguished by the fact that the convergence of a solution of the problem (N) to a reduced solution takes place under the assumption of strong stability either at a boundary point or at an interior point. Namely, in the case of boundary layer behavior we required for example at $t = a$ that if $p_1u(a) - p_2u'(a) \neq A$ then $(p_1u(a) - p_2u'(a) - A)f(a,u(a),\lambda) < 0$ for all λ in $(u'(a),p_2^{-1}(p_1u(a) - A)]$ or $[p_2^{-1}(p_1u(a) - A),u'(a))$. Here $u = u_R$ or u_I. Similarly in the case of interior layer behavior we required that if $u(t_0) = \tilde{u}(t_0) = c$ and $\sigma_L = u'(t_0^-) \neq \tilde{u}'(t_0^+) = \sigma_R$ then $(\sigma_R - \sigma_L)f(t_0,c,\lambda) > 0$ for all λ in (σ_L,σ_R) or (σ_R,σ_L). However it is possible that the same qualitative results are valid if these strict inequalities are replaced by suitable nonstrict ones. We term such phenomena "singular" since they invariably involve the case in which the reduced equation $f = 0$ is singular at one or more points in (a,b) and along various solution trajectories. For example, if $f_{y'}(t_0,y,y') = 0$ for all y, y' of interest then the point t_0 is a singular point of f (cf. [10; Chapter 3]). It will become apparent shortly that the assumption of

y-stability is crucial in obtaining the analogs of the theorems of the previous section. This is not surprising since if $f(t_0,u(t_0),u'(t_0)) = 0$ and $f_{y'}(t_0,u(t_0),u'(t_0)) = 0$ the solution u loses y'-stability in passing through t_0 and so it has to derive stability from the y variable.

Consider first the case of boundary layer behavior. We only state and prove the analog of Theorem 4.1 and then comment on the modifications necessary for proving the analogs of the other boundary layer results.

Theorem 5.1. *Assume that the reduced problem* (R_R) *has a solution* $u = u_R(t)$ *which exists in* $[a,b]$ *and which is weakly* y'-stable *and* y-stable. *Assume also that*

$$(5.1) \qquad (p_1u_R(a) - p_2u_R'(a) - A)f(t,y,y') \leq 0 \quad \text{for} \quad (t,y,y') \quad \text{in} \quad \mathcal{D}(u_R) \cap [a,a+\delta].$$

Then there exists an $\varepsilon_0 > 0$ *such that the problem* (N) *with* $p_2 > 0$ *has a solution* $y = y(t,\varepsilon)$ *whenever* $0 < \varepsilon \leq \varepsilon_0$. *In addition, for* t *in* $[a,b]$ *we have that*

$$y(t,\varepsilon) = u_R(t) + O(w_L(t,\varepsilon)) + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = u_R'(t) + O(w_L'(t,\varepsilon)) + O(\varepsilon),$$

where $w_L(t,\varepsilon) = -(m^{-1}\varepsilon)^{1/2}p_2^{-1}(p_1u_R(a) - p_2u_R'(a) - A)e^{-(m\varepsilon^{-1})^{1/2}(t-a)}$ *is a solution of* $\varepsilon z'' = mz$, $a < t < b$, $z'(a,\varepsilon) = p_2^{-1}(p_1u_R(a) - p_2u_R'(a) - A)$.

Proof. It is only necessary to construct appropriate bounding functions α and β. Define for $a \leq t \leq b$ and $\varepsilon > 0$

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= u_R(t) - \varepsilon\gamma m^{-1} \\ \beta(t,\varepsilon) &= u_R(t) + w_L(t,\varepsilon) + \varepsilon\gamma m^{-1} \end{aligned}\right\} \text{if } \; p_1 u_R(a) - p_2 u_R'(a) \leq A$$

and

$$\left.\begin{aligned} \alpha(t,\varepsilon) &= u_R(t) + w_L(t,\varepsilon) - \varepsilon\gamma m^{-1} \\ \beta(t,\varepsilon) &= u_R(t) + \varepsilon\gamma m^{-1} \end{aligned}\right\} \text{if } \; p_1 u_R(a) - p_2 u_R'(a) \geq A.$$

Consider just the case $p_1 u_R(a) - p_2 u_R'(a) \leq A$. Clearly $\varepsilon\alpha'' \geq f(t,\alpha,\alpha')$ since u_R is y-stable. As for β we have that

$$\begin{aligned} f(t,\beta,\beta') &= f(t,u_R,u_R') + (f(t,\beta,u_R') - f(t,u_R,u_R')) \\ &\quad + \{f(t,\beta,\beta') - f(t,\beta,u_R')\} \\ &= f_y(t,\xi,u_R')(\beta - u_R) + \{\cdot\} \end{aligned}$$

where ξ is the appropriate intermediate point. Therefore

$$f(t,\beta,\beta') - \varepsilon\beta'' = f_y(t,\xi,u_R')(w_L + \varepsilon\gamma m^{-1}) + \{\cdot\} - \varepsilon u_R'' - \varepsilon w_L''$$

$$\geq m w_L + \varepsilon\gamma - \{\cdot\} - \varepsilon M - \varepsilon w_L''$$

$$(M = \max|u_R''|) \geq \varepsilon\gamma - \varepsilon M + \{\cdot\}$$

by our choice of w_L. Now for t in $[a,a + \delta]$ the expression in parentheses is nonnegative by assumption and for t in $[a + \delta,b]$ it is transcendentally small, that is, $\{\cdot\} = \mathcal{O}(\varepsilon^N)$ for all $N \geq 1$. Consequently, if $\gamma = M + 1$ and ε is sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$, we have the desired inequality $\varepsilon\beta'' \leq f(t,\beta,\beta')$ in (a,b).

For the situations described in Theorems 4.2, 4.3, 4.6 and 4.7 the corresponding "singular" analogs are valid if the weakly y'-stable function u_L satisfies

(5.2) $\quad (q_1 u_L(b) + q_2 u_L'(b) - B) f(t,y,y') \leq 0$ for (t,y,y') in $\mathcal{D}(u_L) \cap [b - \delta,b]$,

and if the locally weakly y'-stable function u_I satisfies (5.1) and/or (5.2) with u_R and/or u_L respectively replaced by u_I.

Consider next the case of interior layer behavior. Once again we will only state and prove the analog of Theorem 4.4 and simply indicate the modifications required in the other interior layer results.

Theorem 5.2. Assume that the reduced problems (R_L) and (R_R) have solutions $u = u_L(t)$ and $u = u_R(t)$ in $[a,t_L)$ and $(t_R,b]$ respectively with $t_R < t_L$ such that $u_L(t_0) = u_R(t_0)$ and $\sigma_L = u_L'(t_0) \neq u_R'(t_0) = \sigma_R$ at a point t_0 in (t_R,t_L). Assume also that the path $u = u_1(t)$ is weakly y'-stable and y-stable. and that

$$(5.3) \qquad (\sigma_R - \sigma_L)f(t,y,y') \geq 0 \quad \text{for} \quad (t,y,y') \quad \text{in} \quad \mathcal{D}(u_1) \cap [t_0 - \delta, t_0 + \delta].$$

Then there exists an $\varepsilon_0 > 0$ such that the problem (N) has a solution $y = y(t,\varepsilon)$ whenever $0 < \varepsilon \leq \varepsilon_0$. In addition, we have that

$$y(t,\varepsilon) = u_1(t) + O(w_\ell(t,\varepsilon)) + O(w_r(t,\varepsilon)) + O(\varepsilon) \quad \text{for} \quad a \leq t \leq b,$$

$$y'(t,\varepsilon) = u_L'(t) + O(w_\ell'(t,\varepsilon)) + O(\varepsilon) \qquad \text{for} \quad a \leq t \leq t_0,$$

and

$$y'(t,\varepsilon) = u_R'(t) + O(w_r'(t,\varepsilon)) + O(\varepsilon) \qquad \text{for} \quad t_0 \leq t \leq b.$$

Here $w_\ell(t,\varepsilon) = \frac{1}{2}(\varepsilon m^{-1})^{1/2}(\sigma_R - \sigma_L)e^{(\varepsilon^{-1}m)^{1/2}(t-t_0)}$ is a solution of $\varepsilon z'' = mz$, $a < t < t_0$, $z'(t_0^-,\varepsilon) = \frac{1}{2}(\sigma_R - \sigma_L)$, and $w_r(t,\varepsilon) = \frac{1}{2}(\varepsilon m^{-1})^{1/2}(\sigma_R - \sigma_L) \cdot e^{-(\varepsilon^{-1}m)^{1/2}(t-t_0)}$ is a solution of $\varepsilon z'' = mz$, $t_0 < t < b$, $z'(t_0^+,\varepsilon) = \frac{1}{2}(\sigma_L - \sigma_R)$.

Proof. Suppose for example that $\sigma_L < \sigma_R$ and define for $\varepsilon > 0$

$$\alpha(t,\varepsilon) = u_1(t) - \varepsilon\gamma m^{-1}, \quad a \leq t \leq b,$$

and

$$\beta(t,\varepsilon) = \begin{cases} u_L(t) + w_\ell(t,\varepsilon) + \varepsilon\gamma m^{-1}, & a \le t \le t_0, \\ \\ u_R(t) + w_r(t,\varepsilon) + \varepsilon\gamma m^{-1}, & t_0 \le t \le b. \end{cases}$$

Then it is a straightforward matter to show that for γ sufficiently large and ε sufficiently small, say $0 < \varepsilon \le \varepsilon_0$, these functions satisfy the correct inequalities.

As regards the analogs of Theorems 4,5, 4.6 and 4.7 we must assume that

(5.4) $\quad (\mu_j - \sigma_j)f(t,y,y') \ge 0$ for (t,y,y') in $\mathcal{D}(u) \cap [t_j - \delta, t_j + \delta]$.

Here $j = 1$ and/or 2 and $u = u_2$, u_3 or u_4.

We close this section with two remarks.

Remark 5.1. In discussing certain problems it is necessary to amend condition (5.1) as follows. First of all, if $u_R'' \ge 0$ (≤ 0) in $[a,a + \delta]$ and $p_1u_R(a) - p_2u_R'(a) < A$ $(> A)$ then the proof of Theorem 5.1 shows that it is enough to assume in place in (5.1) that

(5.1)' $\quad f(t,u_R(t),\lambda) \ge 0$ (≤ 0) for t in $[a,a + \delta]$

and for all λ in $[p_2^{-1}(p_1u_R(a) - A), u_R'(a))$ $((u_R'(a), p_2^{-1}(p_1u_R(a) - A)])$. Secondly, if in the original condition (5.1) $(p_1u_R(a) - p_2u_R'(a) - A)f(t,y,y') \le \mu(\varepsilon)$ for a positive function μ which is such that $\mu(\varepsilon) \le L(\varepsilon)\varepsilon$ with $L(\varepsilon) = O(1)$

depending only on ε, then the conclusion of Theorem 5.1 remains valid. Similar remarks apply to the condition (5.2).

It is often necessary to also amend the condition (5.3). If $u_L'' \geq 0$ (≤ 0) and $u_R'' \geq 0$ (≤ 0) in $[t_0 - \delta, t_0]$ and $[t_0, t_0 + \delta]$ respectively and if $\sigma_L < \sigma_R$ ($\sigma_L > \sigma_R$) then the conclusion of Theorem 5.2 is valid if (5.3) is replaced by

(5.3)' $\qquad f(t,u_1(t),\lambda) \geq 0$ (≤ 0) for t in $[t_0 - \delta, t_0 + \delta]$ and for all λ in (σ_L, σ_R) $((\sigma_R, \sigma_L))$.

Secondly, if in (5.3) $(\sigma_R - \sigma_L)f(t,y,y') \leq \mu(\varepsilon)$ with μ as before then the conclusion of Theorem 5.2 is also valid. The conditions (5.4) can be modified in a similar manner.

Remark 5.2. If we assume in the theorems of this section that the reduced solutions are y-stable in a sense more general than that given in Definition 4.8 then the layer corrector terms w must be modified accordingly (cf. for example [9]). The qualitative results are nevertheless the same.

6. Some Examples

We close the paper with several examples that illustrate the theory in Sections 4 and 5.

Example 6.1. Consider the problem

$$\text{(E4)} \qquad \begin{aligned} &\varepsilon y'' = y - ty' - y'^3 = f(t,y,y'), \quad -1 < t < 1, \\ &-y'(-1,\varepsilon) = A, \quad y(1,\varepsilon) = B. \end{aligned}$$

Note that solutions of (E4) are unique by the maximum principle (cf. [16]). The reduced equation $u = tu' + u'^3$ is a Clairaut equation (cf. [10; Chapter 3]) whose solutions are the straight lines $u = u(t) = ct + c^3$ and their envelope $u = \pm \frac{2}{3\sqrt{3}}(-t)^{3/2}$ which is a singular solution defined for $t \leq 0$; see Figure 1.

Figure 1.

Suppose first that $B = 2$. Then the straight line $u = u_R(t) = t + 1$ is a solution of the reduced problem (R_R) corresponding to (E4) which is strongly y'-stable in $[-1,1]$ since $f_{y'}[u_R(t)] = -t - 3 \leq -2$ there. In order to apply Theorem 4.1 we must determine for what values of A

$$(-u_R'(-1) - A)f(-1,u_R(-1),\lambda) = (-1 - A)\lambda(1 - \lambda^2) < 0$$

for all λ in $(u_R'(-1) = 1,-A]$ or $[-A,1)$. If $A = -1$ then $y(t,\varepsilon) = u_R(t)$ is the solution of (E4). If $A > -1$ then

$$(-1 - A)\lambda(1 - \lambda^2) < 0 \quad \text{for} \quad \lambda \quad \text{in} \quad [-A,1)$$

provided $-A > 0$, that is, for $A < 0$. Similarly, if $A < -1$ then

$$(-1 - A)\lambda(1 - \lambda^2) < 0 \quad \text{for} \quad \lambda \quad \text{in} \quad (1,-A]$$

provided $-A > 1$, that is, for $A < -1$. Thus by Theorem 4.1 the problem (E4) for $A < 0$ has a solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) \to u_R(t) = t + 1$ in $[-1,1]$ and $y'(t,\varepsilon) \to 1$ in $(-1,1]$ as $\varepsilon \to 0^+$.

Suppose next that $A = 0$ and $B = 5/8$. Then $u = u_L \equiv 0$ is a solution of the reduced problem (R_L) and $u = u_R(t) = \frac{1}{2}t + \frac{1}{8}$ is a solution of (R_R) which intersect at $t_0 = -\frac{1}{4}$. The corresponding angular path $u = u_1(t)$ is strongly y'-stable since $f_{y'}[u_L(t)] = -t \geq \frac{1}{4}$ in $[-1,-\frac{1}{4}]$ and $f_{y'}[u_R(t)] = -t - \frac{3}{4} \leq -\frac{1}{2}$ in $[-\frac{1}{4},1]$. To apply Theorem 4.4 we must check the condition

that $(\sigma_R - \sigma_L)f(-\frac{1}{4},0,\lambda) > 0$ for λ between σ_L and σ_R, that is,

$$\frac{1}{2}f(-\frac{1}{4},0,\lambda) = \frac{1}{2}\lambda(\frac{1}{4} - \lambda^2) > 0$$

for λ in $(0,\frac{1}{2})$. Thus we deduce from Theorem 4.4 that the problem (E4) has a solution $y = y(t,\varepsilon)$ such that

$$y(t,\varepsilon) \to u_1(t) = \begin{cases} 0, & -1 \le t \le -\frac{1}{4}, \\ \frac{1}{2}t + \frac{1}{8}, & -\frac{1}{4} \le t \le 1, \end{cases}$$

and

$$y'(t,\varepsilon) \to \begin{cases} 0, & -1 \le t < -\frac{1}{4}, \\ \frac{1}{2}, & -\frac{1}{4} < t \le 1, \end{cases} \quad \text{as } \varepsilon \to 0^+.$$

We consider finally two applications of Theorem 4.6. Set $B = \frac{10}{27}$. The unique solution u of the reduced equation $f = 0$ satisfying $u(1) = B$ is $u = u_R(t) = \frac{1}{3}t + \frac{1}{27}$ and it intersects the lower branch u_I $(= -\frac{2}{3\sqrt{3}}(-t)^{3/2})$ of the singular solution at the point $t_2 = -\frac{1}{3}$ (cf. Figure 1). Since u_I is a singular we know also that $u_I'(t_2) = u_R'(t_2)$ and so the corresponding reduced path $u = u_3(t)$ is of class $C^{(1)}[-1,1]$. It remains for us to determine the values of A for which

$$(-u_I'(-1) - A)f(-1,u_I(-1),\lambda) = (-\frac{1}{\sqrt{3}} - A)(-\frac{2}{3\sqrt{3}} + \lambda - \lambda^3) < 0$$

for λ in $(\frac{1}{\sqrt{3}}, -A]$ or $[-A,\frac{1}{\sqrt{3}})$. Note that if $A = -\frac{1}{\sqrt{3}}$ then $y(t,\varepsilon) \to u_3(t)$ and $y'(t,\varepsilon) \to u_3'(t)$ in $[-1,1]$ as $\varepsilon \to 0^+$. If however $-\frac{1}{\sqrt{3}} - A > 0$ then

$$(-\frac{1}{\sqrt{3}} - A)(-\frac{2}{3\sqrt{3}} + \lambda - \lambda^3) = (\frac{1}{\sqrt{3}} + A)(\lambda - \frac{1}{\sqrt{3}})^2(\lambda + \frac{2}{\sqrt{3}}) < 0$$

for all λ in $(\frac{1}{\sqrt{3}},-A]$, while if $-\frac{1}{\sqrt{3}} - A < 0$ then $(\frac{1}{\sqrt{3}} + A)(\lambda - \frac{1}{\sqrt{3}})^2(\lambda + \frac{2}{\sqrt{3}}) \nless 0$ for λ in $[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}})$. Consequently we can apply Theorem 4.6 only if $A \leq -\frac{1}{\sqrt{3}}$ to conclude that the problem (E4) has a solution $y = y(t,\varepsilon)$ such that $y(t,\varepsilon) \to u_3(t)$ in $[-1,1]$ and $y'(t,\varepsilon) \to u_3'(t)$ in $(-1,1]$ as $\varepsilon \to 0^+$.

Suppose finally that $A = \frac{1}{3}$ and $B = \frac{26}{27}$. Then the unique solution u of $f = 0$ satisfying $u(1) = B$ is $u = u_R(t) = \frac{2}{3}t + \frac{8}{27}$ and it intersects the upper branch $u_I(= \frac{2}{3\sqrt{3}}(-t)^{3/2})$ of the singular solution at $t_2 = -\frac{1}{3}$. For this choice of A and B note that $\sigma_L = u_I'(-\frac{1}{3}) = -\frac{1}{3} < u_R'(-\frac{1}{3}) = \frac{2}{3} = \sigma_R$ in contrast to the previous problem. We first check the condition for a crossing at t_2, namely

$$(\sigma_R - \sigma_L)f(-\frac{1}{3},\frac{2}{27},\lambda) = \frac{2}{27} + \frac{1}{3}\lambda - \lambda^3 > 0$$

for λ in $(-\frac{1}{3},\frac{2}{3})$. But

$$-\lambda^3 + \frac{1}{3}\lambda + \frac{2}{27} = -(\lambda + \frac{1}{\sqrt{3}})^2(\lambda - \frac{2}{3}) > 0$$

for such λ and so there is a crossing at t_2. Thus Theorem 4.6 tells us that there is a solution $y = y(t,\varepsilon)$ of (E4) such that

$$y(t,\varepsilon) \to u_2(t) = \begin{cases} u_I(t), & -1 \leq t \leq -\frac{1}{3}, \\ u_R(t), & -\frac{1}{3} \leq t \leq 1, \end{cases}$$

and

$$y'(t,\varepsilon) \to \begin{cases} u_I'(t), & -1 \leq t < -\frac{1}{3}, \\ u_R'(t), & -\frac{1}{3} < t \leq 1, \end{cases} \quad \text{as } \varepsilon \to 0^+.$$

<u>Example 6.2</u>. Consider now the problem

$$\text{(E5)} \quad \begin{aligned} &\varepsilon y'' = y + ty' + y'^n = f(t,y,y'), \quad -1 < t < 1, \\ &y(-1,\varepsilon) - y'(-1,\varepsilon) = A, \quad y(1,\varepsilon) + y'(1,\varepsilon) = B, \end{aligned}$$

for n an integer greater than two, which we will use to illustrate Theorem 4.3. Once again solutions of (E5) are unique by the maximum principle. The function $u_I \equiv 0$ is clearly a solution of the reduced equation $f = 0$ which is locally strongly y'-stable since $f_{y'}[0] = t$. Suppose first that n is

odd. In order to apply Theorem 4.3 we must consider inequalities at $t = -1$ <u>and</u> $t = 1$, namely

(6.1) $(u_I(-1) - u_I'(-1) - A)f(-1,u_I(-1),\lambda) < 0$ for λ in $(u_I'(-1), U_I(-1) - A]$ or $[u_I(-1) - A,u_I'(-1))$

and

(6.2) $(u_I(1) + u_I'(1) - B)f(1,u_I(1),\lambda) < 0$ for λ in $(u_I'(1),B - u_I(1)]$ or $[B - u_I(1),u_I'(1))$.

Condition (6.1) is equivalent to

$$-A\lambda(1 - \lambda^{n-1}) < 0 \quad \text{for } \lambda \text{ in } (0,-A] \text{ or } [-A,0)$$

and this is satisfied for $|A| < 1 (A \neq 0)$ since n is odd. On the other hand, condition (6.2) is equivalent to

$$-B\lambda(1 + \lambda^{n-1}) < 0 \quad \text{for } \lambda \text{ in } (0,B] \text{ or } [B,0)$$

which is true for all $B \neq 0$. If $A = B = 0$ then $y(t,\varepsilon) \equiv 0$ is the solution of (E5) and so from Theorem 4.3 we deduce that if n is odd and $|A| < 1$ then for all values of B the problem (E5) has a solution $y = y(t,\varepsilon)$ such that

$$(6.3) \qquad y(t,\varepsilon) \to 0 \text{ in } [-1,1] \text{ and } y'(t,\varepsilon) \to 0 \text{ in } (-1,1) \text{ as } \varepsilon \to 0^{+}.$$

If now n is even then condition (6.1) is clearly satisfied by all values of $A > -1$ $(A \neq 0)$ while condition (6.2) is satisfied by all values of $B > -1$ $(B \neq 0)$. Thus from Theorem 4.3 we deduce that if n is even and $A,\ B > -1$ then the problem (E5) has a solution $y = y(t,\varepsilon)$ satisfying the limiting relations (6.3).

Example 6.3. Consider next the problem

$$(E6) \qquad \begin{aligned} &\varepsilon y'' = ty'^3 + y^3 - y = f(t,y,y'), \quad -1 < t < 1, \\ &y(-1,\varepsilon) - y'(-1,\varepsilon) = A, \quad y(1,\varepsilon) + y'(1,\varepsilon) = B. \end{aligned}$$

We will show that for certain choices of A and B this problem has at least two solutions.

The reduced equation $f = 0$ has many solutions but we single out just the constant ones $u_1 \equiv 1$ and $u_2 \equiv -1$ which are y-stable since $f_y[\pm 1] = 2$. Note also that both u_1 and u_2 are locally weakly y'-stable since $f_{y'} = 3ty'^2$. We consider only u_1 in detail since the corresponding results for u_2 follow by reflection $(y \to -y)$. To apply Theorem 4.3 we must check the two inequalities:

$$(6.4) \qquad (u_1(-1) - u_1'(-1) - A)f(-1,u_1(-1),\lambda) < 0 \text{ for } \lambda \text{ in}$$
$$(u_1'(-1),u_1(-1) - A] \text{ or } [u_1(-1) - A,u_1'(-1));$$

(6.5) $\quad (u_1(1) + u_1'(1) - B)f(1,u_1(1),\lambda) < 0$ for λ in

$(u_1'(1), B - u_1(1)]$ or $[B - u_1(1), u_1'(1))$.

Condition (6.4) is equivalent to

$$(1 - A)\lambda^3 > 0 \text{ for } \lambda \text{ in } (0,1 - A] \text{ or } [1 - A,0),$$

which is satisfied for all $A \neq 1$. Similarly condition (6.5) is equivalent to

$$(1 - B)\lambda^3 < 0 \text{ for } \lambda \text{ in } (0,B - 1] \text{ or } [B - 1,0),$$

which is satisfied for all $B \neq 1$. Thus by Theorem 4.3 the problem (E6) has a solution $y = y_1(t,\varepsilon)$ such that for all A and B $y_1(t,\varepsilon) \to 1$ in $[-1,1]$ and $y_1'(t,\varepsilon) \to 0$ in $(-1,1)$ as $\varepsilon \to 0^+$. Consequently this problem has another solution $y = y_2(t,\varepsilon)$ such that for all A and B $y_2(t,\varepsilon) \to -1$ in $[-1,1]$ and $y_2'(t,\varepsilon) \to 0$ in $(-1,1)$ as $\varepsilon \to 0^+$.

Example 6.4. In this final example we illustrate some of the singular phenomena discussed in Section 5. The problem is

$$\varepsilon y'' = y - ty'^3 = f(t,y,y'), \quad a < t < b,$$

(E7)

$$-y'(a,\varepsilon) = A, \quad q_1 y(b,\varepsilon) + q_2 y'(b,\varepsilon) = B,$$

whose solutions are unique by the maximum principle. Suppose first that $a = q_2 = B = 0$ and $b = q_1 = 1$, and consider the function $u = u_R(t) \equiv 0$. Clearly u_R is a solution of $f = 0$ satisfying $u_R(1) = B$ which is locally weakly y'-stable since $f_{y'} = -3ty'^2$. Now $f(0,0,0) = 0$ and so we cannot apply Theorem 4.1 but we suspect that for all values of A there is a solution $y = y(t,\varepsilon)$ of (E7) such that

$$(6.6) \qquad y(t,\varepsilon) \to 0 \text{ in } [0,1] \text{ and } y'(t,\varepsilon) \to 0 \text{ in } (0,1] \text{ as } \varepsilon \to 0^+.$$

To establish this we note that condition (5.1)' (cf. Remark 5.1) of Theorem 5.1 holds, namely

$$(-u_R'(0) - A)f(t,u_R(t),\lambda) = At\lambda^3 \leq 0$$

for t in $[0,\delta]$ and λ in $(0,-A]$ or $[-A,0)$. Thus by Theorem 5.1 the problem (E7) has a solution $y = y(t,\varepsilon)$ satisfying the limiting relations (6.6) for all values of A.

Suppose next that $a = -1$, $A = q_1 = q_2 = 1$ and $B = 2$, and consider the functions $u = u_L(t) = -t$ and $u = u_R(t) = t$. Clearly u_L is a solution of the corresponding reduced problem (R_L) while u_R is a solution of (R_R). These functions intersect at $t_0 = 0$ and the angular path $u = u_1(t) = |t|$ is weakly y'-stable since $f_{y'} = -3ty'^2$. However Theorem 4.4 is inapplicable because $f(0,0,\lambda) \equiv 0$ for all λ. We are led to consider applying Theorem 5.2 since u_1 is y-stable and so we have to verify condition (5.3)' (cf. Remark 5.1), that is,

$$f(t,u_1(t),\lambda) \geq 0 \quad \text{for} \quad |t| \leq \delta \quad \text{and} \quad |\lambda| < 1.$$

For t in $[-\delta,0] f(t,u_1(t),\lambda) = -t(1 + \lambda^3) \geq 0$ and for t in $[0,\delta] f(t,u_1(t),\lambda)$ $= t(1 - \lambda^3) \geq 0$. Therefore Theorem 5.2 tells us that the problem (E7) has a solution $y = y(t,\varepsilon)$ such that

$$y(t,\varepsilon) \to |t| \quad \text{in} \quad [-1,1]$$

and

$$y'(t,\varepsilon) \to \begin{cases} -1, & -1 \leq t < 0, \\ 1, & 0 < t \leq 1, \end{cases} \quad \text{as} \quad \varepsilon \to 0^+.$$

Acknowledgment. Portions of this paper were written at the University of Southern California and the University of Arizona and the author wishes to thank Professors W. A. Harris, Jr. and R. E. O'Malley, Jr. for their hospitality. He also wishes to thank the NSF and the ONR for their financial support.

References

1. E. A. Coddington and N. Levinson, A Boundary Value Problem for a Nonlinear Differential Equation with a Small Parameter, Proc. Amer. Math. Soc. 3 (1952), 73-81.

2. F. W. Dorr, S. V. Parter and L. F. Shampine, Applications of the Maximum Principle to Singular Perturbation Problems, SIAM Rev. 15 (1973), 43-88.

3. L. E. El'sgol'c, Qualitative Methods in Mathematical Analysis, American Math. Soc., Providence, R. I., 1964.

4. A. Erdélyi, Singular Perturbations of Boundary Value Problems Involving Ordinary Differential Equations, SIAM J. Appl. Math. 11 (1963), 105-116.

5. S. Haber and N. Levinson, A Boundary Value Problem for a Singularly Perturbed Differential Equation, Proc. Amer. Math. Soc. 6 (1955), 866-872.

6. W. Hahn, Stability of Motion, Springer-Verlag, New York, 1967.

7. J. W. Heidel, A Second-Order Nonlinear Boundary Value Problem, J. Math. Anal. Appl. 48 (1974), 493-503.

8. F. A. Howes, An Asymptotic Theory for a Class of Nonlinear Robin Problems, J. Differential Eqns., to appear.

9. F. A. Howes, Singularly Perturbed Superquadratic Boundary Value Problems, J. Nonlinear Analysis, to appear.

10. E. L. Ince, Ordinary Differential Equations, Dover, New York, 1956.

11. L. K. Jackson, Subfunctions and Second-Order Ordinary Differential Inequalities, Adv. in Math. 2 (1968), 307-363.

12. W. E. Johnson and L. M. Perko, Interior and Exterior Boundary Value Problems from the Theory of the Capillary Tube, Arch. Rational Mech. Anal. 29 (1968), 125-143.

13. R. E. O'Malley, Jr., On Singular Perturbation Problems with Interior Nonuniformities, J. Math. Mech. 19 (1970), 1103-1112.

14. L. M. Perko, Boundary Layer Analysis of the Wide Capillary Tube, Arch. Rational Mech. Anal. 45 (1972), 120-133.

15. L. M. Perko, Singularly Perturbed Two-Point Boundary Value Problems with $(y')^n$, $n > 2$, Nonlinearities, preprint, 1978.

16. M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

17. A. B. Vasil'eva, Asymptotic Behavior of Solutions to Certain Problems Involving Nonlinear Differential Equations Containing a Small Parameter Multiplying the Highest Derivatives, Russian Math. Surveys 18 (1963), 13-84.

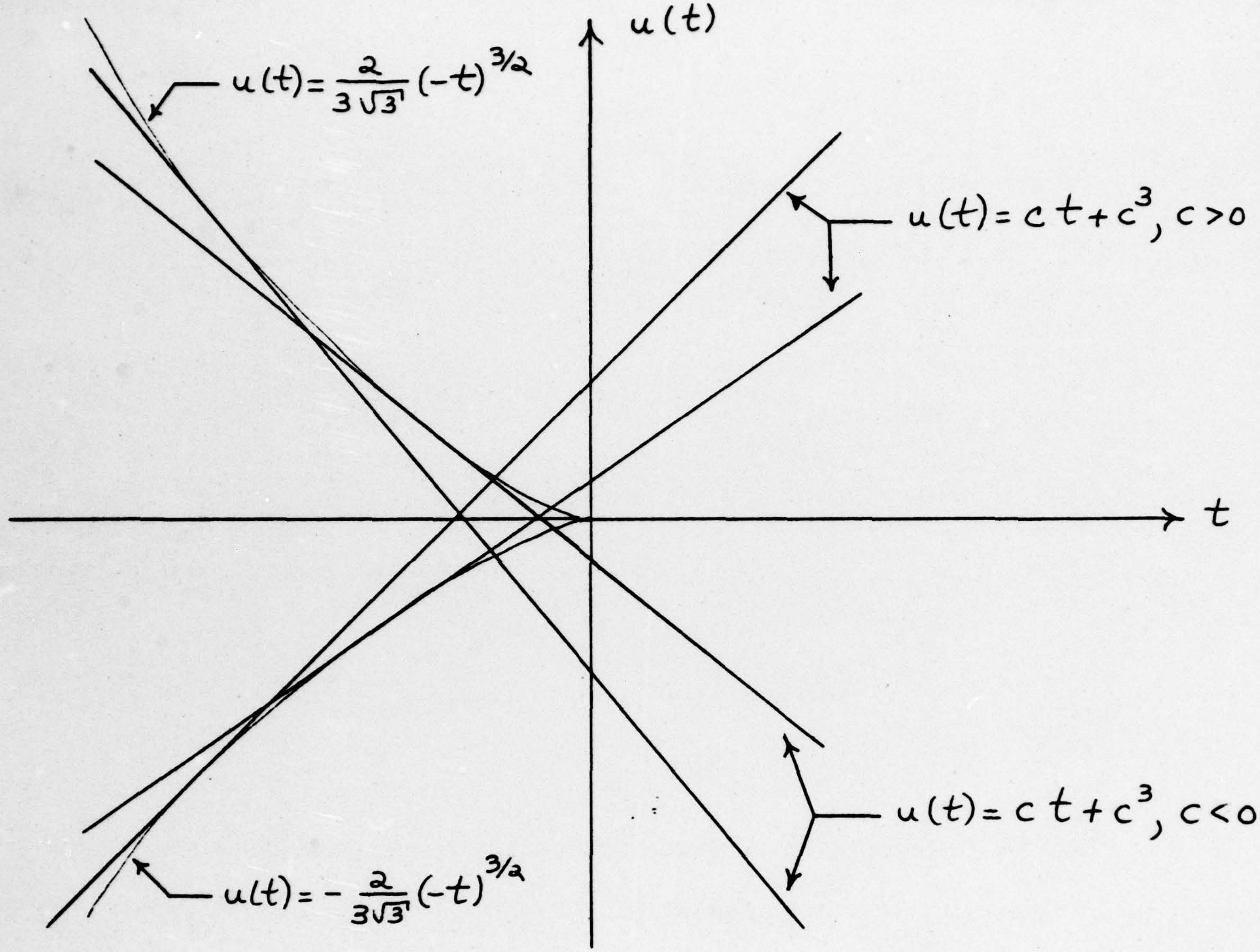

FIGURE 1

SOLUTIONS OF THE CLAIRAUT EQUATION $u = tu' + u'^3$

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Twenty-Four	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) An Asymptotic Theory for a Class of Nonlinear Robin Problems, II		5. TYPE OF REPORT & PERIOD COVERED Technical Report August 1978
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) F. A. Howes		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0326
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics University of Arizona Tucson, Arizona 85721		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR041-466
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Branch Office of Naval Research Arlington, Virginia		12. REPORT DATE August 1978
		13. NUMBER OF PAGES Fifty-Nine (59)
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release. Distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Singularly perturbed boundary value problem, Robin boundary condition, y'-stable solution, y-stable solution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The existence and the asymptotic behavior of solutions of a class of singularly perturbed second-order Robin boundary value problems are studied by means of nonlinear stability theory. Several examples are discussed in detail.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)